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Singüleriteye Sahip Sturm-Liouville Operatörler İçin İz (Trace) Formülü

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Özet: $L_2[0,\pi]$ uzayında $-\frac{d^2}{dx^2}+q(x)$ operatörünün Dirichlet spektrumu $\mu_1,\mu_2,\cdots,\mu_n,\cdots$ olsun. Özel durum olarak $q(x)\equiv 0$ olduğunda $\mu_n = n^2$ dir. I.M. Gelfand, B.M. Levitan [1] ve diğerleri, $q(x)\in C_2[0,\pi]$ için

$$\mu_n = n^2 + \frac{1}{\pi}\int_0^\pi q(x)dx + \mathrm{O}(n^{-2})$$

asimptotik formülü ve olduğunda

$$\sum_n [\mu_n - n^2] = \frac{q(\pi)+q(0)}{4}$$

iz formülünü bulmuşlardır. Bunlar ters (inverse) problemlerin çözümü gibi birçok alanda uygulaması olan formüllerdir. Bu çalışmada, yukarıda ele alınan problem noktasında $\left(\frac{\ell(\ell-1)}{x^2}+\frac{A}{x}\right)$ $x=0$ .singüleriteye sahip Sturm-Liouville operatorü için çalışılmıştır.

Anahtar Kelimeler: İz formülü, spectrum.

Trace formula For Sturm-Liouville Operator With Singularity

Abstract: Let $\mu_1, \mu_2, \cdots, \mu_n, \cdots$ be the Dirichlet spectrum of the operator $-\frac{d^2}{dx^2} + q(x)$ acting on $L_2[0,\pi]$. Inthe special case where $q(x) \equiv 0, \mu_n = n^2$ In the I.M. Gelfand, B.M. Levitan [1] and others discovered the asymptotic formula

$$\mu_n = n^2 + \frac{1}{\pi}\int_0^\pi q(x)dx + O(n^{-2})$$

and the trace formula

$$\sum_n [\mu_n - n^2] = \frac{q(\pi) + q(0)}{4}$$

provided that $\int_0^\pi q(x)dx = 0$, where $q(x) \in C_2[0,\pi]$. These are beatiful formulas with many applications for example in solving inverse problems. In this work, the above mentioned problem has been studied for a Sturm-Liouville operator with $\left(\frac{\ell(\ell-1)}{x^2} + \frac{A}{x}\right)$ singularity at $x = 0$.

Keyword: Trace formula, spectrum.

I. Giriş

Kuantum teorisinde Coulomb potansiyelli alanda elektronlarının hareketlerinin incelenmesi büyük önem taşımaktadır. Bu tip problemlerin çözümü sadece Hidrojen atomunun değil, bir valentli atoma sahip Sodyum ve benzeri atomlarında spektrumunun ve enerji seviyelerinin bulunmasını sağlar. İndirgenmiş m kütle li bir parçacığın 3-boyutlu uzayda $V(r) = -Ze^2/r$ potansiyelindeki hareketi için radyal Schrödinger denklemi,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left[E + \frac{Ze^2}{r} - \frac{\ell(\ell-1)\hbar^2}{2mr^2}\right]R(r) = 0$$

şeklindedir. Burada $\hbar$ Planck sabiti, m parçacığın kütlesidir. λ enerjiye karşılık gelen spektral parametre olmak üzere gerekli dönüşümler yapıldığında,

$$-y'' + \left\{\frac{\ell(\ell-1)}{x^2} + \frac{A}{x} + q(x)\right\}y = \lambda y$$

şeklinde Strum-Liouville denklemi elde edilir. Bu tip singüler diferansiyel ifadelerin tanım kümesinden olan fonksiyonlar için $y'(0)$ sonlu değeri mevcut değildir.

$W_2^2[0,\pi]$ uzayında

$$l(y) := -y'' + \left\{ \frac{\ell(\ell-1)}{x^2} + \frac{A}{x} + q(x) \right\} y \tag{1.1}$$

diferansiyel ifadesinin ve

$$y(0) = 0 \tag{1.2}$$

$$y'(\pi) - Hy(\pi) = 0 \tag{1.3}$$

sınır koşullarının ürettiği operatör L olsun. Burada $q(x) \in W_2^2[0,\pi]$, $\ell \geq 1$ tamsayı, A, H gerçel sayılardır. L operatörünün tanım kümesi:

$D(L) = \{y(x) : y(x) \in W_2^2[0,\pi], l(y) \in L_2[0,\pi], q(x) \in W_2^2[0,\pi], y(0) = 0, y'(\pi) - Hy(\pi) = 0\}$

L operatörünün özdeğerleri ve özfonksiyonları R.Kh. Amirov ve S. Gülyaz çalışmasında [3] verilmiştir.

$A : H \to H$ bir lineer operatör olsun. H bir Hilbert uzayıdır. Hilbert uzayı olduğu için $\{e_n\}_{n\geq 1}$ ve $\|e_n\|_H = 1$ ortonormal sistemi seçilebilir ve $Ae_n = \lambda_n e_n$ olur. Buna göre,

$$\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle = \sum_{n=1}^{\infty} \langle \lambda_n e_n, e_n \rangle = \sum_{n=1}^{\infty} \lambda_n \langle e_n, e_n \rangle = \sum_{n=1}^{\infty} \lambda_n$$

$\sum_{n=1}^{\infty} \lambda_n < +\infty$ ise bu seriye A operatörünün **izi** denir ve $tr(A) := \sum_{n=1}^{\infty} \lambda_n$ ile gösterilir.

$L : L_2[0,\pi] \to L_2[0,\pi], L_2[0,\pi]$ Hilbert uzayı olduğu için $\{y_n\}_{n\geq 0}$ ortonormal sistemi seçilebilir ve $Ly_n = \lambda_n y_n$ olur. Buna göre,

$$\sum_{n=1}^{\infty} \langle Ly_n, y_n \rangle = \sum_{n=1}^{\infty} \langle \lambda_n y_n, y_n \rangle = \sum_{n=1}^{\infty} \lambda_n \langle y_n, y_n \rangle = \sum_{n=1}^{\infty} \lambda_n$$

şeklindedir. Ancak bu seri yakınsak değildir. Dolayısıyla regülerize edilmiş izi incelenecektir. μ_n ile $q(x) = 0$ olduğu duruma karşılık gelen (1.1)-(1.3) probleminin özdeğerlerini göstersin.

Tanım 1.1: $\sum_{n=1}^{\infty} [\lambda_n - \mu_n]$ ifadesine (1.1)-(1.3) probleminin **regülerize edilmiş izi** denir.

2. L Operatörünün Regülerize İzinin Hesaplanması

$\varphi(x,\rho)$ ile

$$l[\varphi(x,\rho)] = \rho^2 \varphi(x,\rho), \quad \lambda =: \rho^2$$

diferansiyel ifadesinin

$$\varphi(0,\rho)=0$$

koşulunu sağlayan çözümü gösterilsin. Bu durumda $\varphi(x,\rho)$ için

$$\varphi(x,\rho)=\sqrt{\frac{\pi\rho x}{2}}J_\nu(\rho x)+$$

$$+(-1)^\ell\frac{\pi}{2}\int_0^x\sqrt{xt}\left[J_\nu(\rho t)J_{-\nu}(\rho x)-J_{-\nu}(\rho t)J_\nu(\rho x)\right]\left(\frac{A}{t}+q(t)\right)\varphi(t,\rho)dt$$

integral denklemi elde edilir.

R.Kh. Amirov ve S. Gülyaz çalışmasında [3], n'nin yeterince büyük değerlerinde L operatörünün özdeğerleri için

$$\rho_n=n+\frac{\ell}{2}+\frac{A}{2\pi}\frac{\ln(n+\ell/2)}{n+\ell/2}+\frac{a_0}{n+\ell/2}-\frac{A^2}{8}\frac{\ln(n+\ell/2)}{(n+\ell/2)^2}+\frac{a_1}{(n+\ell/2)^2}-$$

$$-\frac{A^3}{24\pi}\frac{\ln^3(n+\ell/2)}{(n+\ell/2)^3}+a_2\frac{\ln^2(n+\ell/2)}{(n+\ell/2)^3}+a_3\frac{\ln(n+\ell/2)}{(n+\ell/2)^3}+O\left(\frac{1}{n^3}\right) \tag{2.1}$$

davranışına sahip olduğu gösterilmiştir.

μ_n'ler ile

$$-y''+\left\{\frac{\ell(\ell-1)}{x^2}+\frac{A}{x}-\mu\right\}y=0$$

denklemi ve (1.2)-(1.3) sınır koşulları tarafından üretilen operatörün özdeğerleri gösterilsin.

O halde ρ_n'lerin (2.1) davranışından görüldüğü gibi $(\lambda_1-\mu_1)+(\lambda_2-\mu_2)+\cdots$ serisi $\int_0^\pi q(t)dt=0$ koşulu sağlandığında yakınsaktır.

$\psi(x,\rho)$ fonksiyonu

$$-\psi''(x,\rho)+\left\{\frac{\ell(\ell-1)}{x^2}+\frac{A}{x}+q(x)\right\}\psi(x,\rho)=-\rho^2\psi(x,\rho)$$

denkleminin

$$\psi(0,\rho)=0$$

koşulunu sağlayan çözümü ve $\psi_0(x,\rho)$ ise bu problemin $q(x)\equiv 0$ durumuna karşılık gelen çözümü olsun.

M.G. Gasimov çalışmasında [2]

$$\sum_{n=1}^{\infty}[\lambda_n - \mu_n] = -\lim_{\rho\to i\infty}\frac{\rho^3}{2}\frac{d}{d\rho}\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right) \tag{2.2}$$

eşitliği gösterilmiştir. (1.1.)-(1.3) probleminin regülerize izini hesaplamak için (2.2) eşitliğinden yararlanılacaktır. (2.2) eşitliğinden yararlanmak için $\psi'(\pi,\rho)-H\psi(\pi,\rho)$ ve $\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)$ ifadelerinin $\rho\to i\infty$ iken davranışları bilinmelidir. Bunun için $\psi(x,\rho)$ ve $\psi_0(x,\rho)$ fonksiyonlarının sağladığı integral denklemlerden yararlanılır.

O halde $\ell = 2k+1$ için

$$\psi(x,\rho) = -i\sqrt{\rho x}e^{-i\nu\pi/2}J_\nu(i\rho x)+$$

$$+(-1)^\ell\frac{\pi}{2}\int_0^x\sqrt{xt}\left[J_\nu(i\rho t)J_{-\nu}(i\rho x)-J_{-\nu}(i\rho t)J_\nu(i\rho x)\right]\left(\frac{A}{t}+q(t)\right)\psi(t,\rho)dt$$

olduğundan gerekli işlemler yapıldığında $\psi(x,\rho)$ fonksiyonu için $\rho\to i\infty$ iken

$$\psi(x,\rho) = (-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\left\{\sinh\rho x+\frac{A}{2}\frac{\cosh\rho x}{\rho}\ln\rho x+\frac{A\pi}{4}\frac{\sinh\rho x}{\rho}+\right.$$

$$+\frac{\cosh\rho x}{2\rho}\left[\int_0^x q(t)dt+\frac{A}{2}\left(4+4N_1'-N_2'+N_3'\right)\right]-\frac{\sinh\rho x}{4\rho^2}\left[q(x)+q(0)\right]+$$

$$+\frac{A}{4x}\frac{\sinh 3\rho x}{\rho^2}+\frac{A}{8x^2}\frac{\cosh 3\rho x}{\rho^3}+\frac{A\nu^2}{4x^2}\frac{\cosh\rho x}{\rho^3}+\frac{\cosh\rho x}{8\rho^3}\left[q'(x)-q'(0)\right]-$$

$$\left.-\frac{A\nu^2}{2x^2}\frac{\cosh\rho x}{\rho^3}\cosh 2\rho x+O\left(\frac{1}{\rho^4}\right)\right\}$$

elde edilir. Bu durumda $\psi'(x,\rho)$ fonksiyonu için $\rho\to i\infty$ iken

$$\psi'(x,\rho) = (-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\left\{\rho\cosh\rho x+\frac{A}{2}\sinh\rho x\ln\rho x+\frac{A\pi}{4}\cosh\rho x+\right.$$

$$+\frac{\sinh\rho x}{2}\left[\int_0^x q(t)dt+\frac{A}{2}\left(4+4N_1'-N_2'+N_3'\right)\right]+\frac{\cosh\rho x}{4\rho}\left[q(x)-q(0)\right]+$$

$$+\frac{A}{2}\frac{\cosh\rho x}{\rho x}+\frac{3A}{4x}\frac{\cosh 3\rho x}{\rho}-\frac{A}{4x^2}\frac{\sinh\rho x}{\rho}+\frac{3A}{8x^2}\frac{\sinh 3\rho x}{\rho^2}+$$

$$+\frac{A\nu^2}{4x^2}\frac{\sinh\rho x}{\rho^2}+\frac{\sinh\rho x}{8\rho^2}\left[q'(x)-q'(0)\right]-\frac{A\nu^2}{2x^2}\frac{\sinh\rho x}{\rho^2}\cosh 2\rho x-$$

$$-\frac{A\nu^2}{x^2}\frac{\cosh\rho x}{\rho^2}\sinh 2\rho x-\frac{A}{4x^3}\frac{\cosh 3\rho x}{\rho^3}+\frac{A\nu^2}{2x^3}\frac{\cosh\rho x}{\rho^3}+\frac{\cosh\rho x}{8\rho^3}q''(x)+$$

$$+\frac{A\nu^2}{x^3}\frac{\cosh\rho x}{\rho^3}\cosh 2\rho x+\mathrm{O}\left(\frac{1}{\rho^4}\right)\Bigg\}$$

davranışı bulunur.

$$\psi'(\pi,\rho)-H\psi(\pi,\rho)=(-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\rho\Bigg\{\cosh\rho\pi+\frac{A}{2}\frac{\sinh\rho\pi}{\rho}\ln\rho\pi+\frac{A\pi}{4}\frac{\cosh\rho\pi}{\rho}+$$

$$+M_1'\frac{\sinh\rho\pi}{\rho}-H\frac{\sinh\rho\pi}{\rho}-\frac{AH}{2}\frac{\cosh\rho\pi}{\rho^2}\ln\rho\pi-\frac{AH\pi}{4}\frac{\sinh\rho\pi}{\rho^2}+\frac{A}{2\pi}\frac{\cosh\rho\pi}{\rho^2}+$$

$$+\frac{\cosh\rho\pi}{4\rho^2}\left[q(\pi)-q(0)\right]+\frac{3A}{4\pi}\frac{\cosh 3\rho\pi}{\rho^2}-HM_1'\frac{\cosh\rho\pi}{\rho^2}+\frac{A}{8\pi^2}\frac{\sinh 3\rho\pi}{\rho^3}+$$

$$+\frac{A\nu^2}{4\pi^2}\frac{\sinh\rho\pi}{\rho^3}+\frac{\sinh\rho\pi}{8\rho^3}\left[q'(\pi)-q'(0)\right]-\frac{A\nu^2}{2\pi^2}\frac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi-$$

$$-\frac{A\nu^2}{\pi^2}\frac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi+H\frac{\sinh\rho\pi}{4\rho^3}\left[q(\pi)-q(0)\right]-\frac{AH}{4\pi}\frac{\sinh 3\rho\pi}{\rho^3}+\mathrm{O}\left(\frac{1}{\rho^4}\right)\Bigg\}$$

Burada $M_1'=\frac{1}{2}\left[\int_0^\pi q(t)dt+\frac{A}{2}\left(4+4N_1'-N_2'+N_3'\right)\right]$ ve $M'=\frac{A}{4}\left(4+4N_1'-N_2'+N_3'\right)$ dir.

Benzer şekilde

$$\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)=(-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\rho\Bigg\{\cosh\rho\pi+\frac{A}{2}\frac{\sinh\rho\pi}{\rho}\ln\rho\pi+\frac{A\pi}{4}\frac{\cosh\rho\pi}{\rho}+$$

$$+\frac{AM'}{4}\frac{\sinh\rho\pi}{\rho}-H\frac{\sinh\rho\pi}{\rho}-\frac{AH}{2}\frac{\cosh\rho\pi}{\rho^2}\ln\rho\pi-\frac{AH\pi}{4}\frac{\sinh\rho\pi}{\rho^2}+\frac{A}{2\pi}\frac{\cosh\rho\pi}{\rho^2}+$$

$$+\frac{3A}{4\pi}\frac{\cosh 3\rho\pi}{\rho^2}-\frac{AM'H}{4}\frac{\cosh\rho\pi}{\rho^2}+\frac{A}{8\pi^2}\frac{\sinh 3\rho\pi}{\rho^3}+\frac{A\nu^2}{4\pi^2}\frac{\sinh\rho\pi}{\rho^3}-$$

$$-\frac{A\nu^2}{2\pi^2}\frac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi-\frac{A\nu^2}{\pi^2}\frac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi-\frac{AH}{4\pi}\frac{\sinh 3\rho\pi}{\rho^3}+\mathrm{O}\left(\frac{1}{\rho^4}\right)\Bigg\}$$

elde edilir.

$$\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}=1+$$

$$+\frac{\dfrac{\cosh\rho\pi}{4\rho^2}\left[q(\pi)-q(0)\right]+}{\cosh\rho\pi+\dfrac{A}{2}\dfrac{\sinh\rho\pi}{\rho}\ln\rho\pi+\dfrac{A\pi}{4}\dfrac{\cosh\rho\pi}{\rho}+\left(\dfrac{AM'}{4}-H\right)\dfrac{\sinh\rho\pi}{\rho}-\dfrac{AH}{2}\dfrac{\cosh\rho\pi}{\rho^2}\ln\rho\pi-}$$

$$\frac{+\dfrac{\sinh\rho\pi}{8\rho^3}\left[q'(\pi)-q'(0)\right]+}{-\dfrac{AH\pi}{4}\dfrac{\sinh\rho\pi}{\rho^2}+\dfrac{A}{2\pi}\dfrac{\cosh\rho\pi}{\rho^2}+\dfrac{3A}{4\pi}\dfrac{\cosh 3\rho\pi}{\rho^2}-\dfrac{AM'H}{4}\dfrac{\cosh\rho\pi}{\rho^2}+\dfrac{A}{8\pi^2}\dfrac{\sinh 3\rho\pi}{\rho^3}+} \tag{2.3}$$

$$\frac{+\dfrac{H}{4}\dfrac{\sinh\rho\pi}{\rho^3}\left[q(\pi)+q(0)\right]+O\left(\dfrac{1}{\rho^4}\right)}{+\dfrac{A\nu^2}{4\pi^2}\dfrac{\sinh\rho\pi}{\rho^3}-\dfrac{AH}{4\pi}\dfrac{\sinh 3\rho\pi}{\rho^3}-\dfrac{A\nu^2}{2\pi^2}\dfrac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi-\dfrac{A\nu^2}{\pi^2}\dfrac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi+O\left(\dfrac{1}{\rho^4}\right)}$$

ifadesi bulunur.

Şimdi ise (2.2) eşitliğinin sağ tarafındaki ifadede bulunan

$$\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right)$$

fonksiyonunun $\rho\to i\infty$ iken davranışı araştırılır. (2.3) ifadesinden yararlanarak;

$$\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right)=\ln\left(1+\frac{B_1(\rho)}{C_1(\rho)}\right)$$

$$=\frac{1}{4\rho^2}\left[q(\pi)-q(0)\right]+\frac{1}{8\rho^3}\left[q'(\pi)-q'(0)\right]+\frac{H}{4\rho^3}\left[q(\pi)+q(0)\right]+\frac{e^{-2\rho\pi}}{4\rho^3}\left[q(\pi)-q(0)\right]-$$

$$-\frac{e^{-2\rho\pi}}{8\rho^3}\left[q'(\pi)-q'(0)\right]-\frac{H}{4}\frac{e^{-2\rho\pi}}{\rho^3}\left[q(\pi)+q(0)\right]-\frac{A}{8}\frac{\ln\rho\pi}{\rho^3}\left[q(\pi)-q(0)\right]-$$

$$-\frac{A\pi}{16\rho^3}\left[q(\pi)-q(0)\right]-\left(\frac{AM'}{4}-H\right)\frac{1}{4\rho^3}\left[q(\pi)-q(0)\right]+\frac{A}{8}\frac{e^{-2\rho\pi}}{\rho^3}\ln\rho\pi\left[q(\pi)-q(0)\right]-$$

$$-\frac{A\pi}{16}\frac{e^{-2\rho\pi}}{\rho^3}\left[q(\pi)-q(0)\right]+\left(\frac{AM'}{4}-H\right)\frac{e^{-2\rho\pi}}{4\rho^3}\left[q(\pi)-q(0)\right]+O\left(\frac{\ln\rho}{\rho^4}\right)$$

elde edilir. Burada,

$$B_1(\rho):=\frac{\cosh\rho\pi}{4\rho^2}\left[q(\pi)-q(0)\right]+\frac{\sinh\rho\pi}{8\rho^3}\left[q'(\pi)-q'(0)\right]+$$

$$+\frac{H}{4}\frac{\sinh\rho\pi}{\rho^3}\left[q(\pi)+q(0)\right]+O\left(\frac{1}{\rho^4}\right)$$

$$C_1(\rho) := \cosh\rho\pi + \frac{A}{2}\frac{\sinh\rho\pi}{\rho}\ln\rho\pi + \frac{A\pi}{4}\frac{\cosh\rho\pi}{\rho} + \left(\frac{AM'}{4} - H\right)\frac{\sinh\rho\pi}{\rho} - \frac{AH\pi}{4}\frac{\sinh\rho\pi}{\rho^2} -$$

$$-\frac{AH}{2}\frac{\cosh\rho\pi}{\rho^2}\ln\rho\pi + \frac{A}{2\pi}\frac{\cosh\rho\pi}{\rho^2} + \frac{3A}{4\pi}\frac{\cosh 3\rho\pi}{\rho^2} - \frac{AM'H}{4}\frac{\cosh\rho\pi}{\rho^2} + \frac{A}{8\pi^2}\frac{\sinh 3\rho\pi}{\rho^3} +$$

$$+\frac{A\nu^2}{4\pi^2}\frac{\sinh\rho\pi}{\rho^3} - \frac{AH}{4\pi}\frac{\sinh 3\rho\pi}{\rho^3} - \frac{A\nu^2}{2\pi^2}\frac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi - \frac{A\nu^2}{\pi^2}\frac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi + \mathrm{O}\left(\frac{1}{\rho^4}\right)$$

dir. Dolayısıyla

$$\frac{d}{d\rho}\ln\left(\frac{\psi'(\pi,\rho) - H\psi(\pi,\rho)}{\psi_0'(\pi,\rho) - H\psi_0(\pi,\rho)}\right) = -\frac{\pi e^{-2\rho\pi}}{2\rho}\left[q(\pi) - q(0)\right] - \frac{A\pi e^{-2\rho\pi}}{4\rho^2}\ln\rho\pi\left[q(\pi) - q(0)\right] +$$

$$+\frac{\pi e^{-2\rho\pi}}{4\rho^2}\left[q'(\pi) - q'(0)\right] + \frac{H}{2}\frac{\pi e^{-2\rho\pi}}{\rho^2}\left[q(\pi) + q(0)\right] + \frac{A\pi^2}{8}\frac{e^{-2\rho\pi}}{\rho^2}\left[q(\pi) - q(0)\right] -$$

$$-\left(\frac{AM'}{4} - H\right)\frac{\pi e^{-2\rho\pi}}{2\rho^2}\left[q(\pi) - q(0)\right] - \frac{1}{2\rho^3}\left[q(\pi) - q(0)\right] + \mathrm{O}\left(\frac{\ln\rho}{\rho^4}\right)$$

bulunur. Buradan

$$-\frac{\rho^3}{2}\frac{d}{d\rho}\ln\left(\frac{\psi'(\pi,\rho) - H\psi(\pi,\rho)}{\psi_0'(\pi,\rho) - H\psi_0(\pi,\rho)}\right) = \frac{\pi}{4}\rho^2 e^{-2\rho\pi}\left[q(\pi) - q(0)\right] +$$

$$+\frac{A\pi}{8}\rho e^{-2\rho\pi}\ln\rho\pi\left[q(\pi) - q(0)\right] - \frac{\pi}{8}\rho e^{-2\rho\pi}\left[q'(\pi) - q'(0)\right] -$$

$$-\frac{H\pi}{4}\rho e^{-2\rho\pi}\left[q(\pi) + q(0)\right] - \frac{A\pi^2}{16}\rho e^{-2\rho\pi}\left[q(\pi) + q(0)\right] +$$

$$+\left(\frac{AM'}{4} - H\right)\frac{\pi}{4}\rho e^{-2\rho\pi}\left[q(\pi) - q(0)\right] + \frac{\left[q(\pi) - q(0)\right]}{4} + \mathrm{O}\left(\frac{1}{\rho^4}\right)$$

olduğundan $\int_0^{\pi} q(t)dt = 0$ koşulu sağlandığında

$$-\lim_{\rho\to i\infty}\frac{\rho^3}{2}\frac{d}{d\rho}\ln\left(\frac{\psi'(\pi,\rho) - H\psi(\pi,\rho)}{\psi_0'(\pi,\rho) - H\psi_0(\pi,\rho)}\right) = \frac{q(\pi) - q(0)}{4}$$

eşitliği elde edilir. O halde

$$\sum_{n=1}^{\infty}[\lambda_n - \mu_n] = \frac{q(\pi) - q(0)}{4}$$

olur.

Şimdi benzer işlemler $\ell = 2k+1$ için yapılırsa

$$\psi(x,\rho)=i\sqrt{\rho x}e^{-i\nu\pi/2}J_{-\nu}(i\rho x)+$$

$$+\frac{\pi}{2}\int_0^x\sqrt{xt}\left[J_\nu(i\rho t)J_{-\nu}(i\rho x)-J_{-\nu}(i\rho t)J_\nu(i\rho x)\right]\left(\frac{A}{t}+q(t)\right)\psi(t,\rho)dt$$

olduğundan birinci bölümde kullanılan yöntem uygulanarak $\psi(x,\rho)$ fonksiyonu için $\rho\to i\infty$ iken

$$\psi(x,\rho)=-(-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\left\{\sinh\rho x+\frac{A}{2}\frac{\cosh\rho x}{\rho}\ln\rho x-\frac{A\pi}{4}\frac{\sinh\rho x}{\rho}+\right.$$

$$+\frac{\cosh\rho x}{2\rho}\left[-\int_0^x q(t)dt+\frac{AM'}{2}\right]+\frac{\sinh\rho x}{4\rho^2}\left[q(x)+q(0)\right]+\frac{A}{4x}\frac{\sinh 3\rho x}{\rho^2}+$$

$$+\frac{A}{8x^2}\frac{\cosh 3\rho x}{\rho^3}+\frac{A\nu^2}{4x^2}\frac{\cosh\rho x}{\rho^3}-\frac{\cosh\rho x}{8\rho^3}\left[q'(x)-q'(0)\right]-$$

$$\left.-\frac{A\nu^2}{2x^2}\frac{\cosh\rho x}{\rho^3}\cosh 2\rho x+\mathrm{O}\left(\frac{1}{\rho^4}\right)\right\}$$

elde edilir. Bu durumda $\psi'(x,\rho)$ fonksiyonu için $\rho\to i\infty$ iken

$$\psi'(x,\rho)=-(-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\left\{\rho\cosh\rho x+\frac{A}{2}\sinh\rho x\ln\rho x-\frac{A\pi}{4}\cosh\rho x+\right.$$

$$+\frac{\sinh\rho x}{2}\left[-\int_0^x q(t)dt+\frac{AM'}{2}\right]+\frac{A}{2x}\frac{\cosh\rho x}{\rho}-\frac{\cosh\rho x}{4\rho}\left[q(x)-q(0)\right]+$$

$$+\frac{3A}{4x}\frac{\cosh 3\rho x}{\rho}-\frac{A}{4x^2}\frac{\sinh\rho x}{\rho}+\frac{3A}{8x^2}\frac{\sinh 3\rho x}{\rho^2}+\frac{A\nu^2}{4x^2}\frac{\sinh\rho x}{\rho^2}-$$

$$-\frac{\sinh\rho x}{8\rho^2}\left[q'(x)-q'(0)\right]-\frac{A\nu^2}{2x^2}\frac{\sinh\rho x}{\rho^2}\cosh 2\rho x-\frac{A\nu^2}{x^2}\frac{\cosh\rho x}{\rho^2}\sinh 2\rho x-$$

$$-\frac{A}{4x^3}\frac{\cosh 3\rho x}{\rho^3}-\frac{A\nu^2}{2x^3}\frac{\cosh\rho x}{\rho^3}-\frac{\cosh\rho x}{8\rho^3}q''(x)+$$

$$\left.+\frac{A\nu^2}{x^3}\frac{\cosh\rho x}{\rho^3}\cosh 2\rho x+\mathrm{O}\left(\frac{1}{\rho^4}\right)\right\}$$

davranışı bulunur.

$$\psi'(\pi,\rho)-H\psi(\pi,\rho)=-(-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\rho\left\{\cosh\rho\pi+\frac{A}{2}\frac{\sinh\rho\pi}{\rho}\ln\rho\pi-\frac{A\pi}{4}\frac{\cosh\rho\pi}{\rho}+\right.$$

$$+M_2'\frac{\sinh\rho\pi}{\rho}-H\frac{\sinh\rho\pi}{\rho}-\frac{AH}{2}\frac{\cosh\rho\pi}{\rho^2}\ln\rho\pi+\frac{AH\pi}{4}\frac{\sinh\rho\pi}{\rho^2}+\frac{A}{2\pi}\frac{\cosh\rho\pi}{\rho^2}-$$

$$-\frac{\cosh\rho\pi}{4\rho^2}\left[q(\pi)-q(0)\right]+\frac{3A}{4\pi}\frac{\cosh 3\rho\pi}{\rho^2}-HM_2'\frac{\cosh\rho\pi}{\rho^2}+\frac{A}{8\pi^2}\frac{\sinh 3\rho\pi}{\rho^3}+$$

$$+\frac{A\nu^2}{4\pi^2}\frac{\sinh\rho\pi}{\rho^3}-\frac{\sinh\rho\pi}{8\rho^3}\left[q'(\pi)-q'(0)\right]-\frac{A\nu^2}{2\pi^2}\frac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi-$$

$$-\frac{A\nu^2}{\pi^2}\frac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi-H\frac{\sinh\rho\pi}{4\rho^3}\left[q(\pi)-q(0)\right]-\frac{AH}{4\pi}\frac{\sinh 3\rho\pi}{\rho^3}+\mathrm{O}\left(\frac{1}{\rho^4}\right)\Bigg\}$$

Burada $M_2'=\dfrac{AM'}{2}-\int\limits_0^x q(t)dt$ dir.

Benzer şekilde

$$\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)=(-1)^k e^{-i\nu\pi/2}\sqrt{\frac{2}{\pi i}}\rho\Bigg\{\cosh\rho\pi+\frac{A}{2}\frac{\sinh\rho\pi}{\rho}\ln\rho\pi-\frac{A\pi}{4}\frac{\cosh\rho\pi}{\rho}+$$

$$+\frac{AM'}{4}\frac{\sinh\rho\pi}{\rho}-H\frac{\sinh\rho\pi}{\rho}-\frac{AH}{2}\frac{\cosh\rho\pi}{\rho^2}\ln\rho\pi+\frac{AH\pi}{4}\frac{\sinh\rho\pi}{\rho^2}+\frac{A}{2\pi}\frac{\cosh\rho\pi}{\rho^2}+$$

$$+\frac{3A}{4\pi}\frac{\cosh 3\rho\pi}{\rho^2}-\frac{AM'H}{4}\frac{\cosh\rho\pi}{\rho^2}+\frac{A}{8\pi^2}\frac{\sinh 3\rho\pi}{\rho^3}+\frac{A\nu^2}{4\pi^2}\frac{\sinh\rho\pi}{\rho^3}-$$

$$-\frac{A\nu^2}{2\pi^2}\frac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi-\frac{A\nu^2}{\pi^2}\frac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi-\frac{AH}{4\pi}\frac{\sinh 3\rho\pi}{\rho^3}+\mathrm{O}\left(\frac{1}{\rho^4}\right)\Bigg\}$$

elde edilir.

$$\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}=1+$$

$$+\frac{-\dfrac{\cosh\rho\pi}{4\rho^2}\left[q(\pi)-q(0)\right]-}{\cosh\rho\pi+\dfrac{A}{2}\dfrac{\sinh\rho\pi}{\rho}\ln\rho\pi-\dfrac{A\pi}{4}\dfrac{\cosh\rho\pi}{\rho}+\left(\dfrac{AM'}{4}-H\right)\dfrac{\sinh\rho\pi}{\rho}-\dfrac{AH}{2}\dfrac{\cosh\rho\pi}{\rho^2}\ln\rho\pi+}$$

$$\frac{-\dfrac{\sinh\rho\pi}{8\rho^3}\left[q'(\pi)-q'(0)\right]-}{+\dfrac{AH\pi}{4}\dfrac{\sinh\rho\pi}{\rho^2}+\dfrac{A}{2\pi}\dfrac{\cosh\rho\pi}{\rho^2}+\dfrac{3A}{4\pi}\dfrac{\cosh 3\rho\pi}{\rho^2}-\dfrac{AM'H}{4}\dfrac{\cosh\rho\pi}{\rho^2}+\dfrac{A}{8\pi^2}\dfrac{\sinh 3\rho\pi}{\rho^3}+}$$

(2.4)

$$-\frac{H}{4}\frac{\sinh\rho\pi}{\rho^3}\left[q(\pi)+q(0)\right]+O\left(\frac{1}{\rho^4}\right)$$

$$+\frac{A\nu^2}{4\pi^2}\frac{\sinh\rho\pi}{\rho^3}-\frac{AH}{4\pi}\frac{\sinh 3\rho\pi}{\rho^3}-\frac{A\nu^2}{2\pi^2}\frac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi-\frac{A\nu^2}{\pi^2}\frac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi+O\left(\frac{1}{\rho^4}\right)$$

ifadesi bulunur.

Şimdi ise (2.2) eşitliğinin sağ tarafındaki ifadede bulunan

$$\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right)$$

fonksiyonunun $\rho\to i\infty$ iken davranışı araştırılır. (2.4) ifadesinden yararlanarak;

$$\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right)=\ln\left(1+\frac{B_2(\rho)}{C_2(\rho)}\right)$$

$$=-\frac{1}{4\rho^2}\left[q(\pi)-q(0)\right]-\frac{1}{8\rho^3}\left[q'(\pi)-q'(0)\right]-\frac{H}{4\rho^3}\left[q(\pi)+q(0)\right]-\frac{e^{-2\rho\pi}}{4\rho^3}\left[q(\pi)-q(0)\right]+$$

$$+\frac{e^{-2\rho\pi}}{8\rho^3}\left[q'(\pi)-q'(0)\right]+\frac{H}{4}\frac{e^{-2\rho\pi}}{\rho^3}\left[q(\pi)+q(0)\right]+\frac{A}{8}\frac{\ln\rho\pi}{\rho^3}\left[q(\pi)-q(0)\right]-$$

$$-\frac{A\pi}{16\rho^3}\left[q(\pi)-q(0)\right]+\left(\frac{AM'}{4}-H\right)\frac{1}{4\rho^3}\left[q(\pi)-q(0)\right]-\frac{A}{8}\frac{e^{-2\rho\pi}}{\rho^3}\ln\rho\pi\left[q(\pi)-q(0)\right]+$$

$$+\frac{A\pi}{16}\frac{e^{-2\rho\pi}}{\rho^3}\left[q(\pi)-q(0)\right]-\left(\frac{AM'}{4}-H\right)\frac{e^{-2\rho\pi}}{4\rho^3}\left[q(\pi)-q(0)\right]+O\left(\frac{\ln\rho}{\rho^4}\right)$$

elde edilir. Burada,

$$B_2(\rho):=-\frac{\cosh\rho\pi}{4\rho^2}\left[q(\pi)-q(0)\right]-\frac{\sinh\rho\pi}{8\rho^3}\left[q'(\pi)-q'(0)\right]-$$

$$-\frac{H}{4}\frac{\sinh\rho\pi}{\rho^3}\left[q(\pi)+q(0)\right]+O\left(\frac{1}{\rho^4}\right)$$

$$C_2(\rho):=\cosh\rho\pi+\frac{A}{2}\frac{\sinh\rho\pi}{\rho}\ln\rho\pi-\frac{A\pi}{4}\frac{\cosh\rho\pi}{\rho}+\left(\frac{AM'}{4}-H\right)\frac{\sinh\rho\pi}{\rho}+\frac{AH\pi}{4}\frac{\sinh\rho\pi}{\rho^2}-$$

$$-\frac{AH}{2}\frac{\cosh\rho\pi}{\rho^2}\ln\rho\pi+\frac{A}{2\pi}\frac{\cosh\rho\pi}{\rho^2}+\frac{3A}{4\pi}\frac{\cosh 3\rho\pi}{\rho^2}-\frac{AM'H}{4}\frac{\cosh\rho\pi}{\rho^2}+\frac{A}{8\pi^2}\frac{\sinh 3\rho\pi}{\rho^3}+$$

$$+\frac{A\nu^2}{4\pi^2}\frac{\sinh\rho\pi}{\rho^3}-\frac{AH}{4\pi}\frac{\sinh 3\rho\pi}{\rho^3}-\frac{A\nu^2}{2\pi^2}\frac{\sinh\rho\pi}{\rho^3}\cosh 2\rho\pi-\frac{A\nu^2}{\pi^2}\frac{\cosh\rho\pi}{\rho^3}\sinh 2\rho\pi+O\left(\frac{1}{\rho^4}\right)$$

dir. Dolayısıyla

$$\frac{d}{d\rho}\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right)=\frac{\pi e^{-2\rho\pi}}{2\rho}\big[q(\pi)-q(0)\big]+\frac{A\pi e^{-2\rho\pi}}{4\rho^2}\ln\rho\pi\big[q(\pi)-q(0)\big]-$$

$$-\frac{\pi e^{-2\rho\pi}}{4\rho^2}\big[q'(\pi)-q'(0)\big]-\frac{H}{2}\frac{\pi e^{-2\rho\pi}}{\rho^2}\big[q(\pi)+q(0)\big]-\frac{A\pi^2}{8}\frac{e^{-2\rho\pi}}{\rho^2}\big[q(\pi)-q(0)\big]+$$

$$+\left(\frac{AM'}{4}-H\right)\frac{\pi e^{-2\rho\pi}}{2\rho^2}\big[q(\pi)-q(0)\big]+\frac{1}{2\rho^3}\big[q(\pi)-q(0)\big]+\mathrm{O}\left(\frac{\ln\rho}{\rho^4}\right)$$

bulunur. Buradan

$$-\frac{\rho^3}{2}\frac{d}{d\rho}\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right)=-\frac{\pi}{4}\rho^2e^{-2\rho\pi}\big[q(\pi)-q(0)\big]-$$

$$-\frac{A\pi}{8}\rho e^{-2\rho\pi}\ln\rho\pi\big[q(\pi)-q(0)\big]+\frac{\pi}{8}\rho e^{-2\rho\pi}\big[q'(\pi)-q'(0)\big]+$$

$$+\frac{H\pi}{4}\rho e^{-2\rho\pi}\big[q(\pi)+q(0)\big]+\frac{A\pi^2}{16}\rho e^{-2\rho\pi}\big[q(\pi)+q(0)\big]-$$

$$-\left(\frac{AM'}{4}-H\right)\frac{\pi}{4}\rho e^{-2\rho\pi}\big[q(\pi)-q(0)\big]-\frac{\big[q(\pi)-q(0)\big]}{4}+\mathrm{O}\left(\frac{1}{\rho^4}\right)$$

olduğundan $\int_0^\pi q(t)dt=0$ koşulu sağlandığında

$$-\lim_{\rho\to i\infty}\frac{\rho^3}{2}\frac{d}{d\rho}\ln\left(\frac{\psi'(\pi,\rho)-H\psi(\pi,\rho)}{\psi_0'(\pi,\rho)-H\psi_0(\pi,\rho)}\right)=-\frac{q(\pi)-q(0)}{4}$$

eşitliği elde edilir. O halde

$$\sum_{n=1}^{\infty}[\lambda_n-\mu_n]=-\frac{q(\pi)-q(0)}{4}$$

olur. Böylece aşağıdaki teorem ispatlandı:

Teorem: Eğer $\int_0^\pi q(t)dt=0$ ise

$$\sum_{n=1}^{\infty}[\lambda_n-\mu_n]=(-1)^{\ell+1}\frac{q(\pi)-q(0)}{4}$$

eşitliği doğrudur.

Kaynaklar

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