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Gelfand-Levitan-Marchenko Integral Equation for Singular Differential Operators

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Abstract. In this study, the Gelfand-Levitan-Marchenko (GLM) type main integral equation which is important for solution of the inverse problem related to determining of a singular Sturm-Liouville differential operators is obtained.
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1. Introduction

In this paper we consider boundary value problem L for the equation

$$l(y) := -y'' + q(x)y = \lambda y,\ \lambda = k^2 \tag{1.1}$$

on the interval $0 < x < \pi$, with the boundary conditions

$$U(y) := y'(0) = 0,\ V(y) := y(\pi) = 0 \tag{1.2}$$

and with the jump conditions

$$y(d+0) = ay(d-0),\ y'(d+0) = a^{-1}y'(d-0), \tag{1.3}$$

where λ is the spectral parameter, $q(x)$ is a real valued function with $q(x) \in L_2(0,\pi)$ and a $(a > 0, a \neq 1)$ is a real constant, $d \in \left(\frac{\pi}{2}, \pi\right)$.

Inverse spectral analysis has been an important research topic in mathematical physics. Inverse problems of spectral analysis involve the reconstruction of a linear operator from its spectral characteristics e.g., see [2, 15, 24, 25, 30]. A problem of this kind was first investigated by Ambarzumyan in 1929 [3]. Later, inverse problems for a regular and singular Sturm Liouville operator appeared in various versions [3, 5, 6, 9, 14, 15, 17-19, 23, 26, 28, 31-35].

Assuming that heat flows only into the liquid which has an ununiform density $\rho(x)$ and is convected only from the liquid into the surrounding medium, the initial boundary value problem for a bar of length one takes the form

$$u_t = \rho(x)u_{xx} \tag{1*}$$

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$$u_x(0,t) = 0 \tag{2*}$$

$$-kAu_x(\pi,t) = QM(dv/dt) + k_1Bv(t) \text{ for all } t, \tag{3*}$$

$$u(x,0) = u_0(x) \text{ for } x \in [0,\pi], \tag{4*}$$

$$v(0) = v_0$$

after factoring out the steady-state solution, where
$\rho(x) = \begin{cases} 1, & 0 < x < d, \\ \alpha^2, & d < x < \pi. \end{cases}$

Assuming that the rate of heat transfer across the liquid–solid interface is proportional to the difference in temperature between the end of the bar and the liquid with which it is in contact (Newton's law of cooling) and applying Fourier's law of heat conduction at $x = \pi$, we get

$v(t) = u(\pi,t) + kc^{-1}u_x(\pi^{-1},t)$ for $t > 0$, where $c > 0$ is the coefficient of heat transfer for the liquid. If we put $u(x,t) = y(x)exp(-\lambda t)$, then problems (1.1)–(1.3) will appear to be the consequence of the above problem. Indeed, condition (1.2) is obtained from (2*), easily. Here

$H = \dfrac{c}{k}, \qquad H_1 = \dfrac{cA + k_1B}{QM} \qquad \text{and} \qquad H_2 = \dfrac{k_1Bc}{QM_k}.$

Finally, if we put
$t = \begin{cases} x, & 0 < x < d, \\ \alpha x, & d < x < \pi, \end{cases}$
then the discontinuity conditions (1.3) and a particular case of Equation (1.1) will appear. This corresponds to the case of nonperfect thermal contact. Since the density is changed at one point in the interval, both the intensity and the instant velocity of heat change at this point. Hence, Equation (1.1)–(1.3) will appear to be the consequence of the above problem.

Boundary value problems with discontinuity conditions inside the interval often appear in applications. Such problems are connected with discontinuous material properties. Inverse problems with a discontinuity condition inside the interval frequently arise in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [27, 30]. As a rule, such problems are related to discontinuous and nonsmooth properties of a medium (e.g., see [5, 17, 23]). Discontinuous inverse problems (in various formulations) have been considered in [14, 17, 23, 36] and other works. Generally, for recovering the potential function on the whole interval it is necessary to specify two spectra of boundary value problems with different boundary conditions (see [36]). The inverse problem for interior spectral data of the differential operator consists in reconstruction of this operator from the known eigenvalues and some information on eigenfunctions at some internal point. The technique employed is similar to those used in [18, 36].

The solution of the inverse spectral problem for a Sturm-Liouville operator consists the following steps: (1) an explicit description of the spectral data of the considered operator and (2) development and justification the method of recovering the operator corresponding to any given spectral data. The algorithm of recovering the potential q from the spectral data of a regular Sturm-Liouville operator based on the transformation operators and the so-called Gelfand-Levitan-Marchenko equation was developed by Gelfand and Levitan [15] and Marchenko [28] in early 1950-ies.

The first complete solution of the inverse problem that is based on an exact integral approach was obtained by Gelfand and Levitan [1, 10, 13, 15, 21, 34] for the potential problem in the Schrodinger wave equation. In electromagnetics, the above approach is directly applicable to the case of inversion with a transient plane wave, normally incident on a planar stratified lossless medium [15], provided that the wave equation is converted to the Schrodinger equation. Other variations of this classical integral inversion approach have been developed by considering special choices of input-output pairs [7]. Generalizations of the Gel'fand-Levitan approach to the case of oblique incidence [11, 12], dissipative media [22], etc, were all based on deriving a Schrodinger-type equation from the basic wave equation through a series of transformations, and reconstructing the unknown potential, which is related to the medium parameters, via the Gel'fand-Levitan procedure. Other inverse methods which are based on an integral equation and are in the same spirit as the Gel'fand-Levitan approach are the ones due to [7, 10, 13]. A review of some of these integral inverse methods and others can be found in the review paper by Newton [31].

In this aspect, the studies of Gelfand, Levitan [15], [25] and Marchenko[29] include bacis investigations related to constraction of the integral representations for solutions and application them to various direct and inverse problems for Sturm-Liouville differential operators.

In this paper, the Gelfand-Levitan-Marchenko (GLM) type main integral equation which is important for solution of inverse problem related to determining of the Sturm-Liouville differential operators having discontinuity conditions inside a finite interval is investigated.

2. Preliminaries

Let the function $\varphi(x,\lambda)$ be the solution of equation (1.1) that satisfies the initial conditions

$$\varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = 0, \tag{2.1}$$

and the jump condition (1.3). Let $\lambda_0, \lambda_1, ...$be the eigenvalues of the boundary value problem (1.1)-(1.3). Then $\varphi(x,\lambda_n)$ $(n \geq 0)$ are the eigenfunctions of this the boundary value problem. Let $\varphi_0(x,\lambda_n^0)$ $(n \geq 0)$ be a solution of equation (1.1) in the case $q(x) = 0$ satisfying the condition (1.2)-(1.3). $\lambda_0^0, \lambda_1^0, ...$ are eigenvalues of the boundary value problem (1.1)-(1.3) when $q(x) = 0$. The numbers α_n which

$$\alpha_n = \int\limits_0^\pi \varphi^2(x,k_n)dx,\ n = 0, 1, \ldots \tag{2.2}$$

are called the normallizing constant of the boundary value problem (1.1)-(1.3) .

The numbers α_n^0, $(n = 0, 1, \ldots)$ are called the normalizing constant of the boundary value problem (1.1)-(1.3) when $q(x) = 0$.

It is easy to show that in the case $q(x) \equiv 0$ the function $e_0(x,\lambda)$ which is solution of equation (1.1) with initial conditions $e_0(x,\lambda) = 1$, $e_0'(x,\lambda) = ik$ and the jump conditions (1.3) can be written as:

$$e_0(x,\lambda)=\begin{cases} e^{ikx}, & 0<x<d, \\ a^+e^{ikx}+a^-e^{ik(2d-x)}, & d<x<\pi, \end{cases}$$

where $a^{\pm}=\dfrac{1}{2}\left(a\pm\dfrac{1}{a}\right)$.

The following theorem related to the integral representation (transformation operator) for the solution $e(x,\lambda)$ can be found in [4].

Theorem 1. [*4, Theorem 1.*] *Let* $\int\limits_0^{\pi}|q(t)|\,dt<+\infty$. *Then each solution satisfying the initial conditions* $e_0(x,\lambda)=1$, $e_0'(x,\lambda)=ik$ *and the jump conditions (1.3) has the form*

$$e(x,\lambda)=e_0(x,\lambda)+\int\limits_{-x}^{x}K(x,t)e^{ikt}dt$$

with $\int\limits_{-x}^{x}|K(x,t)|\,dt\le e^{c\sigma_1(x)}-1$, *where* $\sigma_1(x)=\int\limits_0^{x}(x-t)\,|q(t)|\,dt$, $c=a^++|a^-|+1$.

If the function $q(x)$ is differentiable then the kernel $K(x,t)$ satisfies the following properties:

$$\widetilde{K}_{xx}(x,t)-q(x)\widetilde{K}(x,t)=\widetilde{K}_{tt}(x,t), \qquad \widetilde{K}(x,x)=\frac{a^+}{2}\int\limits_0^{x}q(t)dt,$$

$$\widetilde{K}(x,2d-x+0)-\widetilde{K}(x,2d-x-0)=\frac{a^-}{2}\int\limits_0^{x}q(t)dt,$$

$\widetilde{K}(x,-x)=0$ where $\widetilde{K}=K(x,t)+K(x,-t)$.

Remark 1. [*4, Remark*] *It is easily shown that if* $q(x)\in L_2\,[0,\pi]$ *then* $K_x(x,.)\in L_2\,[0,\pi]$ *and* $K_t(x,.)\in L_2\,[0,\pi]$.

Let us denote the problem L as L_0 in the case of $q(x)\equiv 0$. It is easily shown that the solution $\varphi_0(x,k)$ satisfying the inital conditions $\varphi_0(0,k)=1$, $\varphi_0'(0,k)=0$ and the jump conditions (1.3) can be written as

$$\varphi_0(x,\lambda)=\begin{cases} \cos kx, & 0<x<d, \\ a^+\cos kx+a^-\cos k(2d-x), & d<x<\pi. \end{cases} \tag{2.3}$$

Let $\Delta_0(k)$ be a characteristic function of problem L_0. Then characteristic equation of the problem L_0can be expressed as

$$\Delta_0(k)\equiv a^+\cos k\pi+a^-\cos k(2d-\pi)=0.$$

The roots k_n^0 of this equation are eigenvalues of the problem L_0.

Lemma 1. [*4, Lemma 1.*] $\inf \left| k_n^0 - k_m^0 \right| = \beta > 0$, *i.e., roots of characteristic equation* $\Delta_0(k) = 0$ *are separated.*

Lemma 2. [*4, Lemma 2.*] *Eigenvaules of the problem* L *are simple, that is*

$$\dot{\Delta}(k_n) \neq 0.$$

Lemma 3. [*4, Lemma 3.*] *Eigenvaules of the problem* L *have the following asymptotic behaviour*

$$k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{\delta_n}{k_n^0}, \tag{2.4}$$

where $\delta_n = \dfrac{1}{k_n^0} \int\limits_0^\pi K_t'(\pi, t) \sin k_n^0 t dt \in \ell_2$, $k_n^0 = n + h_n$, $\sup |h_n| \leq M$ *and*
$d_n = \dfrac{a^+ \sin k_n^0 \pi - a^- \sin k_n^0 (2d - \pi)}{2\dot{\Delta}_0(k_n^0) k_n^0} \int\limits_0^\pi q(t) dt$ *is a bounded sequence.*

Lemma 4. [*4, Lemma 4.*] *Normalizing numbers of the problem* L *have the asymptotic behaviour*

$$\alpha_n = \alpha_n^0 + \delta_n, \tag{2.5}$$

where

$$\alpha_n^0 = ((a^+)^2 + (a^-)^2) \frac{\pi - d}{2} + \frac{d}{2} + 2a^+ a^- (\pi - d) \cos 2k_n^0 d + \delta_{1n} \tag{2.6}$$

and

$$\delta_{1n} = \frac{\sin 2k_n^0 d}{4k_n^0} + (a^+)^2 \frac{\sin 2k_n^0 \pi}{4k_n^0} - (a^+)^2 \frac{\sin 2k_n^0 d}{4k_n^0} + \frac{a^+ a^-}{k_n^0} \sin 2k_n^0 (\pi - d)$$
$$- \frac{(a^-)^2}{4k_n^0} \sin 2k_n^0 (2d - \pi) + \frac{(a^-)^2}{4k_n^0} \sin 2k_n^0 d, \ \delta_n \in \ell_2 \ .$$

3. The Main Integral Equation

In this section, we will obtain the main integral equation for the spectral problem (1.1)-(1.2)-(1.3) which has an important role in recovering the operator. In this reason, we first prove the following Lemma:

Lemma 5. *Assume that numbers* $\{\lambda_n, \alpha_n\}_{n \geq 0}$ *satisfying the conditions of the form (2.4) and (2.5) are given and denote*

$$b(x) := \sum_{n=0}^{\infty} \left(\frac{\cos k_n x}{\alpha_n} - \frac{\cos k_n^0 x}{\alpha_n^0} \right), \tag{3.1}$$

where

$$\alpha_n^0 = \begin{cases} \dfrac{d}{2} + \dfrac{1}{4k_n^0}\sin 2k_n^0 d, & 0 < x < d, \\ \left[(a^+)^2 + (a^-)^2\right]\dfrac{\pi - d}{2} + \dfrac{d}{2} + 2a^+a^-(\pi - d)\cos 2k_n^0 d, & d < x < \pi. \end{cases}$$

Then $b(x) \in W_2^1\,(0,d) \cup (d, 2\pi)$.

Proof. Denote $\varepsilon_n = k_n - k_n^0$. Since

$$\frac{\cos k_n x}{\alpha_n} - \frac{\cos k_n^0 x}{\alpha_n^0} = \frac{1}{\alpha_n^0}\left(\cos k_n x - \cos k_n^0 x\right) + \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0}\right)\cos k_n x,$$

$$\cos k_n x - \cos k_n^0 x = -\varepsilon_n \sin k_n^0 x - \sin k_n^0 x\left(\sin \varepsilon_n x - \varepsilon_n x\right) - 2\sin^2\frac{\varepsilon_n x}{2}\cos k_n^0 x,$$

we have $b(x) = B_1(x) + B_2(x)$, where

$$B_1(x) = -\sum_{n=1}^{\infty} \frac{d_n x \sin k_n^0 x}{\alpha_n^0 k_n^0} \tag{3.2}$$

$$\begin{aligned} B_2(x) = \sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0}\right)\cos k_n x - \sum_{n=1}^{\infty}\frac{\delta_n x \sin k_n^0 x}{\alpha_n^0 k_n^0} \\ - \sum_{n=1}^{\infty}\left(\sin\varepsilon_n x - \varepsilon_n x\right)\frac{\sin k_n^0 x}{\alpha_n^0} - \sum_{n=1}^{\infty} 2\sin^2\frac{\varepsilon_n x}{2}\frac{\cos k_n^0 x}{\alpha_n^0}. \end{aligned} \tag{3.3}$$

Since $\varepsilon_n = O\left(\dfrac{1}{n}\right)$, $\dfrac{1}{\alpha_n} - \dfrac{1}{\alpha_n^0} = -\dfrac{\delta_n}{k_n^0} + O\left(\dfrac{1}{n^3}\right)$, where $\delta_n = \dfrac{1}{k_n^0}\int_0^\pi K_t'(\pi, t)\sin k_n^0 t dt$ the series in (3.2) and (3.3) converge absolutely and uniformly on $(0, d) \cup (d, 2\pi)$ and $B_2(x) \in W_2^1\,(0,d) \cup (d, 2\pi)$, $B_1(x) \in W_2^1\,(0,d) \cup (d, 2\pi)$. Consequently,

$b(x) \in W_2^1\,(0,d) \cup (d, 2\pi)$. □

We will refer to the sequences $\{\lambda_n\}_{n\geq 0}$ and $\{\alpha_n\}_{n\geq 0}$ as the spectral characteristics of the boundary value problem (1.1)-(1.3). Consider the function

$$F(x,t) = \sum_{n=0}^{\infty}\left[\frac{1}{\alpha_n}\varphi_0(x,k_n)\varphi_0(t,k_n) - \frac{1}{\alpha_n^0}\varphi_0(x,k_n^0)\varphi_0(t,k_n^0)\right] \tag{3.4}$$

with the help $\{\lambda_n\}_{n\geq 0}$ and $\{\alpha_n\}_{n\geq 0}$ sequences.

Firstly, we will investigate properties of the function $F(x,t)$ by using the asymptotic expressions for $\varphi_0(x, \lambda_n)$ and $\varphi_0(x, \lambda_n^0)$. Note that, the asymptotic expressions of these functions for sufficiently large values of n are given in [4].

It is clear that if $q(x) = 0$ then the asymptotic formula for $\varphi_0(x, \lambda_n^0)$ is

$$\varphi_0(x, k_n^0) = \begin{cases} \cos k_n^0 x, & 0 < x < d, \\ a^+\cos k_n^0 x + a^-\cos k_n^0(2d - x), & d < x < \pi. \end{cases}$$

Moreover, the asymptotic equalities

$$\frac{1}{\alpha_n}=\frac{1}{\alpha_n^0+\delta_n}=\frac{1}{\alpha_n^0}-\frac{\delta_n}{(\alpha_n^0)^2}+O\left(\frac{1}{n^3}\right)$$

and

$$\frac{1}{\alpha_n^0}=\frac{2}{((a^+)^2+(a^-)^2)\pi+(1-(a^+)^2-(a^-)^2)d}+O\left(\frac{1}{n}\right)$$

are also satisfied.

It is easy to calculate that

(i)- if $0<x<d$ and $0<t<d$ then

$$F(x,t)=\frac{a^+}{2}\left[b(x+t)+b(x-t)\right],$$

(ii)-if $0<x<d$ and $d<t<\pi$ then

$$F(x,t)=\frac{a^+}{2}\left[b(x+t)+b(x-t)\right]+\frac{a^-}{2}\left[b(x+2d-t)+b(x-2d+t)\right],$$

(iii)- if $d<x<\pi$ and $0<t<d$ then

$$F(x,t)=\frac{a^+}{2}\left[b(x+t)+b(x-t)\right]+\frac{a^-}{2}\left[b(2d-x+t)+b(2d-x-t)\right]$$

(iv)- if $d<x<\pi$ and $d<t<\pi$ then

$$F(x,t)=\frac{\left(a^+\right)^2}{2}\left[b(x+t)+b(x-t)\right]+\frac{a^+a^-}{2}\left[b(x+2d-t)+b(x-2d+t)\right]$$

$$+\frac{a^+a^-}{2}\left[b(2d-x+t)+b(2d-x-t)\right]+\frac{\left(a^-\right)^2}{2}\left[b(4d-x-t)+b(t-x)\right].$$

Lemma 5 implies that $F(x,t)$ is continuous and $\frac{d}{dx}F(x,x)\in L_2\left[0,\pi\right]$.

Theorem 2. *[16, Lemma 8]Let $f(x)$, $x\in[0,\pi]$.be an absolutely continuous function. Then*

$$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n}\int_0^{\pi}f(t)\varphi(t,k_n)dt\right)\varphi(x,k_n)=f(x) \tag{3.5}$$

with uniform convergence in $[0,\pi]$

Theorem 3. *For each fixed $x \in (0,\pi]$, the kernel $\tilde{K}(x,t)$ appearing in the representation*

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x \tilde{K}(x,t)\cos kt dt \tag{3.6}$$

satisfies the linear integral equation

$$F(x,t) + a^+\tilde{K}(x,\xi) - a^-\tilde{K}(x,2d-\xi) + \int_0^x \tilde{K}(x,\xi)F_0(\xi,t)d\xi = 0. \tag{3.7}$$

Proof. One can consider the relation (2.3) with respect to $\cos kx$. Solving this equation we obtain

$$\cos kx = \begin{cases} \varphi_0(x,k), & 0 < x < d, \\ a^+\varphi_0(x,k) - a^-\varphi_0(2d-x,k), & d < x < \pi. \end{cases} \tag{3.8}$$

Using equalities (3.6) and (3.8), we calculate

$$\Phi_N(x,t) = \sum_{n=0}^{N}\left(\frac{\varphi(x,k_n)\varphi(t,k_n)}{\alpha_n} - \frac{\varphi_0(x,k_n^0)\varphi_0(t,k_n^0)}{\alpha_n^0}\right)$$
$$= \sum_{n=0}^{N}\left(\frac{\varphi_0(x,k_n)\varphi_0(t,k_n)}{\alpha_n} - \frac{\varphi_0(x,k_n^0)\varphi_0(t,k_n^0)}{\alpha_n^0}\right)$$
$$+ \int_0^x \tilde{K}(x,\xi)\sum_{n=0}^{N}\frac{\varphi_0(t,k_n^0)\cos k_n^0\xi}{\alpha_n^0}d\xi$$
$$+ \int_0^x \tilde{K}(x,\xi)\sum_{n=0}^{\infty}\left(\frac{\varphi_0(t,k_n)\cos k_n\xi}{\alpha_n} - \frac{\varphi_0(t,k_n^0)\cos k_n^0\xi}{\alpha_n^0}\right)d\xi$$
$$+ \int_0^t \tilde{K}(x,\xi)\sum_{n=0}^{N}\frac{\varphi(x,k_n)\cos k_n\xi}{\alpha_n}d\xi.$$

Therefore we can write

$$\Phi_N(x,t) = \Phi_{N_1}(x,t) + \Phi_{N_2}(x,t) + \Phi_{N_3}(x,t) + \Phi_{N_4}(x,t),$$

where

$$\Phi_{N_1}(x,t) = \sum_{n=0}^{N}\left(\frac{\varphi_0(x,k_n)\varphi_0(t,k_n)}{\alpha_n} - \frac{\varphi_0(x,k_n^0)\varphi_0(t,k_n^0)}{\alpha_n^0}\right)$$

$$\Phi_{N_2}(x,t) = \sum_{n=0}^{N}\frac{\varphi_0(t,k_n^0)}{\alpha_n^0}\int_0^x \tilde{K}(x,\xi)\cos k_n^0\xi d\xi$$
$$\Phi_{N_3}(x,t) = \sum_{n=0}^{\infty}\int_0^x \tilde{K}(x,\xi)\left(\frac{\varphi_0(t,k_n)\cos k_n\xi}{\alpha_n} - \frac{\varphi_0(t,k_n^0)\cos k_n^0\xi}{\alpha_n^0}\right)d\xi$$

$$\Phi_{N_4}(x,t) = \sum_{n=0}^{N}\frac{\varphi(x,k_n)}{\alpha_n}\int_0^t \tilde{K}(x,\xi)\cos k_n\xi d\xi.$$

Let $f(x) \in AC[0,\pi]$. Acording to Theorem 2, we obtain (uniformly on $x \in [0,\pi]$):

$$\lim_{N\to\infty} \int_0^\pi f(t)\Phi_N(x,t)dt = 0$$

$$\lim_{N\to\infty} \int_0^\pi f(t)\Phi_{N_1}(x,t)dt = \int_0^\pi f(t)F(x,t)dt$$

$$\lim_{N\to\infty} \int_0^\pi f(t)\Phi_{N_2}(x,t)dt = a^+ \int_0^x f(\xi)\tilde{K}(x,\xi)d\xi + a^- \int_0^x f(\xi)\tilde{K}(x,2d-\xi)d\xi$$

$$\lim_{N\to\infty} \int_0^\pi f(t)\Phi_{N_3}(x,t)dt = \int_0^\pi f(t)\int_0^x \tilde{K}(x,\xi)F_0(\xi,t)d\xi dt$$

$$F_0(x,t) = \sum_{n=0}^{\infty}\left(\frac{\varphi_0(t,k_n)\cos k_n x}{\alpha_n} - \frac{\varphi_0(t,k_n^0)\cos k_n^0 x}{\alpha_n^0}\right)$$

$$F(x,t) = a^+F_0(x,t) + a^-F_0(2d-x,t)$$

$$= \sum_{n=0}^{\infty}\left(\frac{\varphi_0(t,k_n)\varphi_0(x,k_n)}{\alpha_n} - \frac{\varphi_0(t,k_n^0)\varphi_0(x,k_n^0)}{\alpha_n^0}\right)$$

$$\lim_{N\to\infty} \int_0^\pi f(t)\Phi_{N_4}(x,t)dt = \lim_{N\to\infty} \int_0^\pi f(t)\int_0^t \tilde{K}(t,\xi)\sum_{n=0}^{N}\frac{\varphi(x,k_n)\cos k_n\xi}{\alpha_n}d\xi dt.$$

Using $\psi(x,k_n) = \beta_n\varphi(x,k_n)$ and $\alpha_n\beta_n = \dot{\Delta}(k_n)$

$$= -\lim_{N\to\infty} \int_0^\pi f(t)\sum_{|k_n|\le N}\frac{\psi(x,k_n)}{\dot{\Delta}(k_n)}\int_0^t \tilde{K}(t,\xi)\cos k_n\xi d\xi dt$$

$$= -\lim_{N\to\infty} \int_0^\pi f(t)\sum_{|k_n|\le N}\underset{\lambda=\lambda_n}{Res}\left[\frac{\psi(x,k)}{\Delta(k)}\int_0^t \tilde{K}(t,\xi)\cos\lambda\xi d\xi\right]dt$$

$$= -\lim_{N\to\infty} \int_0^\pi f(t)\frac{1}{2\pi i}\oint_{G_N}\frac{\psi(t,k)}{\Delta(k)}\int_0^t \tilde{K}(t,\xi)\cos\lambda\xi d\xi d\lambda dt$$

$$= -\lim_{N\to\infty} \int_0^\pi f(t)\frac{1}{2\pi i}\oint_{G_N}\frac{\psi(t,k)}{\Delta(k)}e^{Imkt}e^{-Imkt}\int_0^t \tilde{K}(t,\xi)\cos\lambda\xi d\xi d\lambda dt$$

$$= -\int_0^\pi f(t)\lim_{N\to\infty}\left[\frac{1}{2\pi i}\oint_{G_N}\frac{\psi(t,k)}{\Delta(k)}e^{Imkt}e^{-Imkt}\int_0^t \tilde{K}(t,\xi)\cos\lambda\xi d\xi d\lambda\right]dt, \qquad 0<t<x,$$

where $G_N := \left\{k : \ |k| = \left|k^0\right| + \frac{\beta}{2},\ n = 0,1,\dots\right\}$,
$G_\delta = \left\{k : \left|k - k_n^0\right| \ge \delta,\ n = 0,1,\dots,\ \delta > 0\right\}$ and δ is sufficiently small positive number$\left(\delta \ll \frac{\beta}{2}\right)$.

The solution $\psi(x,k)$ of equation (1.1) satisfying the conditions $\psi(\pi,k) = 0$, $\psi'(\pi,k) = -1$ and the jump conditions (1.3) is an entire function of λ and

$$\psi(x,k) = O\left(\frac{1}{|k|}e^{|Imk|(\pi-x)}\right),\ |k|\to\infty,\ \ |\Delta(k)| \ge C_\delta\,|k|\,e^{|Imk|\pi},\ \ k \in \overline{G}_\delta,$$

$$\left|\frac{\psi(x,k)}{\Delta(k)}e^{|Imk|t}\right| \leq \frac{C_\delta}{|k|^2}e^{|Imk|(t-x)} \underset{\substack{|k|\to\infty \\ k\in G_\delta}}{\to} 0 \tag{3.9}$$

and

$$\lim_{|k|\to\infty}\max_{0\leq t\leq\pi} e^{-|Imk|t}\left|\int_0^t \tilde{K}(t,\xi)\cos\lambda\xi d\xi\right| = 0. \tag{3.10}$$

Using (3.9) and (3.10), we obtain

$$\lim_{N\to\infty}\int_0^\pi f(t)\Phi_{N_4}(x,t)dt = 0.$$

Hence, we find that

$$\begin{aligned}
&\lim_{N\to\infty}\int_0^\pi f(t)\Phi_N(x,t)dt = \int_0^\pi f(t)F(x,t)dt \\
&+a^+\int_0^x f(\xi)\tilde{K}(x,\xi)d\xi - a^-\int_0^x f(\xi)\tilde{K}(x,2d-\xi)d\xi \\
&+\int_0^\pi f(t)\int_0^x \tilde{K}(x,\xi)F_0(\xi,t)d\xi dt = 0.
\end{aligned}$$

Then, in view of the arbitrariness of $f(x)$, the main integral equation

$$F(x,t) + a^+\tilde{K}(x,\xi) - a^-\tilde{K}(x,2d-\xi) + \int_0^x \tilde{K}(x,\xi)F_0(\xi,t)d\xi = 0$$

is obtained.

When $t < x$ this equation implies (3.7).

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