On Jost Solution of Diffusion Equation With Discontinuous Coefficient

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Abstract. In this work, integral representation of Jost solution of a diffusion equation is obtained and the properties of representation of the kernel are studied.

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1. Introduction

An important role in the spectral theory of linear operators was played by the transformation operator. Marchenko [3] , [5] first applied the transformation operator to the solution of the inverse problem. Transformation operator was also used in the fundamental paper of Gelfand and Levitan [4] . Yurko [11] studied the inverse problem theory for Sturm Liouville operators on the half-line.

$-y'' + q(x)y = \lambda y, \quad x > 0$ has a unique solution satisfying the integral equation

$$e(x,\rho) = \exp(i\rho x) - \frac{1}{2i\rho}\int_x^{+\infty}\left(\exp(i\rho(x-t)) - \exp(i\rho(t-x))\right)q(t)\,e(t,\rho)\,dt.$$

where $\rho \in \left\{\rho : Im\rho \geq 0, \rho \neq 0, \lambda = \rho^2\right\}$. The function $e(x,\rho)$ is called the Jost solution [11] .

Huseynov [14] has considered the differential equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in (0,a) \cup (a,+\infty)$$

with discontinous conditions $y(a-0) = \alpha y(a+0)$, $y'(a-0) = \alpha^{-1}y'(a+0)$, where $\alpha \neq 1, \alpha > 0$, λ is a complex valued function and satisfies the condition $xq(x) \in L^1(I)$. Then for all λ from the upper half-plane, the Jost solution is unique and it is in the form

$$e(x,\lambda) = e_0(x,\lambda) + \int_x^{+\infty} K(x,t)\,e^{i\lambda t}dt.$$

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Let's consider the diffusion equation with discontinous coefficient

$$-y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \rho(x)\, y, \quad x \in (0,a) \cup (a,+\infty) = I. \tag{1.1}$$

we assume that

$$(1+x)\, q(x),\ p(x) \in L^1(I), \tag{1.2}$$

$$p(x) \in BC(I), \tag{1.3}$$

where $L^1(I)$ is the space of integrable functions and $BC(I)$ denotes bounded, continuous functions on I,

λ is a complex parameter and $\rho(x)$ is a step function:

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a \\ \alpha^2, & a < x \le \pi \end{cases}, 0 < \alpha \ne 1. \tag{1.4}$$

The function $e(x,\lambda)$, satisfying the equation (1.1) and condition at infinity $\lim\limits_{x\to\infty} e(x,\lambda)\, e^{-i\lambda x} = 1$ is said to be Jost solution of (1.1). It is easy to show that if $q(x) \equiv 0$, then the Jost solution is the function

$$e_0(x,\lambda) = \begin{cases} R_1(x)\, e^{i\lambda x} & , x > a \\ \alpha^+ R_2(x)\, e^{i\lambda\mu^+(x)} + \alpha^- R_3(x)\, e^{i\lambda\mu^-(x)} & , 0 < x < a \end{cases} \tag{1.5}$$

where $\mu^\pm(x) = \pm\sqrt{\rho(x)}x + a\left(1 \mp \sqrt{\rho(x)}\right)$ and $\alpha^\pm = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$.

2. Main Result

The main result of the present paper is the following.

Theorem. If a complex-valued function $q(x)$ fulfilled condition (1.2), then there exists the Jost solution $e(x,\lambda)$ of equation (1.1) for all λ from the upper half-plane, it is unique and representable in the form

$$e(x,\lambda) = e_0(x,\lambda) + \int_{\mu^+(x)}^{+\infty} K(x,t)\, e^{i\lambda t} dt, \tag{2.1}$$

where, for each fixed $x \in (0,a) \cup (a,+\infty)$, the kernel $K(x,.)$ belongs to the space $L_1(\mu^+(x), \infty)$. Moreover, $K(x,t)$ is countinous with partial derivatives $\frac{\partial K(x,t)}{\partial x}$, $\frac{\partial K(x,t)}{\partial t}$ for $t \ne \mu^-(x)$ and it satisfies the following properties

a) $\int\limits_{\mu^+(x)}^{+\infty} |K(x,t)|\, dt \le e^{\sigma(x)} - 1, \quad \sigma(x) = \int\limits_x^{+\infty} [2|p(t)| + (1+|t|)\, |q(t)|]\, dt \qquad (2.2)$

b) $\alpha^+ R_2''(x) + 2i\lambda\alpha\alpha^+ R_2'(x) - 2\lambda\alpha^+ p(x)\, R_2(x)$

$+2ip(x)\, K(x,\mu^+(x)) - \alpha^+ q(x)\, R_2(x) - 2\alpha \frac{dK(x,\mu^+(x))}{dx} = 0 \qquad (2.3)$

$$c)\alpha^{-}R_3''(x)-2i\lambda\alpha\alpha^{-}R_3'(x)-2\lambda\alpha^{-}p(x)R_3(x)$$
$$-2ip(x)K(x,\mu^{-}(x)-0)+2ip(x)K(x,\mu^{-}(x)+0)$$
$$-\alpha^{-}q(x)R_3(x)-2\alpha\frac{dK(x,\mu^{-}(x)-0)}{dx}+2\alpha\frac{dK(x,\mu^{-}(x)+0)}{dx}=0 \tag{2.4}$$

$$d)\frac{\partial^2K(x,t)}{\partial x^2}-\alpha^2\frac{\partial^2K(x,t)}{\partial t^2}+2ip(x)\frac{\partial K(x,t)}{\partial t}=q(x)K(x,t) \tag{2.5}$$

$$e)\lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial x}=0,\ \lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial t}=0 \tag{2.6}$$

Here $e_0(x,\lambda)$ is in the form of (1.5) which is shown above.
$e(x,\lambda)$ is in the form

$$e(x,\lambda)=e_0(x,\lambda)+\int_x^{+\infty}S_0(x,t,\lambda)\left[2\lambda p(t)+q(t)\right]e(t,\lambda)\,dt, \tag{2.7}$$

where

$$S_0(x,t,\lambda)=\begin{cases}\dfrac{\sin\lambda(x-t)}{\lambda} & ,a<x<t\\ \alpha^{+}\dfrac{\sin\lambda(\mu^{+}(x)-t)}{\lambda}+\alpha^{-}\dfrac{\sin\lambda(\mu^{-}(x)-t)}{\lambda} & ,x<a<t\\ \dfrac{\sin\lambda\alpha(x-t)}{\lambda\alpha} & ,x<t<a\end{cases} \tag{2.8}$$

Proof:
Substituting (2.1) into (2.7) , we get

$$e_0(x,\lambda)+\int_{\mu^{+}(x)}^{+\infty}K(x,t)e^{i\lambda t}dt=e_0(x,\lambda)$$
$$+\int_x^{+\infty}S_0(x,t,\lambda)\left[2\lambda p(t)+q(t)\right]\left(e_0(t,\lambda)+\int_{\mu^{+}(t)}^{+\infty}K(t,\xi)e^{i\lambda\xi}d\xi\right)dt.$$

Firstly, for $x>a$, we arrive

$$\int_x^{+\infty}K(x,t)e^{i\lambda t}dt+(R_1(x)-1)e^{i\lambda x}=$$
$$=\int_x^{+\infty}S_0(x,t,\lambda)(2\lambda p(t)+q(t))\left(R_1(t)e^{i\lambda t}+\int_t^{+\infty}K(t,\xi)e^{i\lambda\xi}d\xi\right)dt$$

Now, if one evaluates the right-hand side of above equation, then it follows that

$$\int_x^{+\infty}S_0(x,t,\lambda)\left[2\lambda p(t)+q(t)\right]R_1(t)e^{i\lambda t}dt$$
$$+\int_x^{+\infty}S_0(x,t,\lambda)\left[2\lambda p(t)+q(t)\right]\left(\int_t^{+\infty}K(t,\xi)e^{i\lambda\xi}d\xi\right)dt$$

$$= \int_x^{+\infty} \sin\lambda(x-t)\, 2p(t)\, R_1(t)\, e^{i\lambda t} dt + \int_x^{+\infty} \frac{\sin\lambda(x-t)}{\lambda} q(t)\, R_1(t)\, e^{i\lambda t} dt$$

$$+ \int_x^{+\infty} \sin\lambda(x-t)\, 2p(t) \left(\int_t^{+\infty} K(t,\xi)\, e^{i\lambda\xi} d\xi \right) dt$$

$$+ \int_x^{+\infty} \frac{\sin\lambda(x-t)}{\lambda} q(t) \left(\int_t^{+\infty} K(t,\xi)\, e^{i\lambda\xi} d\xi \right) dt$$

$$= -i \int_x^{+\infty} p(t)\, R_1(t)\, e^{i\lambda x} dt + i \int_x^{+\infty} p(t)\, R_1(t)\, e^{i\lambda(-x+2t)} dt$$

$$+ \frac{1}{2} \int_x^{+\infty} q(t)\, R_1(t) \left(\int_{-x+2t}^{x} e^{i\lambda u} du \right) dt - i \int_x^{+\infty} p(t) \left(\int_t^{+\infty} K(t,\xi)\, e^{i\lambda(x-t+\xi)} d\xi \right) dt$$

$$+ i \int_x^{+\infty} p(t) \left(\int_t^{+\infty} K(t,\xi)\, e^{i\lambda(-x+t+\xi)} d\xi \right) dt$$

$$+ \frac{1}{2} \int_x^{+\infty} q(t) \left(\int_t^{+\infty} K(t,\xi) \int_{-x+t+\xi}^{x-t+\xi} e^{i\lambda u} du d\xi \right) dt$$

$$= -ie^{i\lambda x} \int_x^{+\infty} p(t)\, R_1(t)\, dt + \frac{i}{2} \int_x^{+\infty} p\left(\frac{x+t}{2}\right) R_1\left(\frac{x+t}{2}\right) e^{i\lambda t} dt$$

$$- \frac{1}{2} \int_x^{+\infty} \left(\int_{(x+t)/2}^{+\infty} q(u)\, R_1(u)\, du \right) e^{i\lambda t} dt$$

$$- i \int_x^{+\infty} \left(\int_x^{+\infty} p(u)\, K(u, t-x+u)\, du \right) e^{i\lambda t} dt$$

$$+ i \int_x^{+\infty} \left(\int_x^{(x+t)/2} p(u)\, K(u, t+x-u)\, du \right) e^{i\lambda t} dt$$

$$+ \frac{1}{2} \int_x^{+\infty} \left(\int_x^{+\infty} q(u) \int_{t+x-u}^{t-x+u} K(u,\xi)\, d\xi du \right) e^{i\lambda t} dt.$$

Thus, in the case of $x > a$, one can obtain the following equations

$$R_1(x) = 1 - i \int_x^{+\infty} p(t)\, R_1(t)\, dt$$

$$K(x,t) = \frac{i}{2} p\left(\frac{x+t}{2}\right) R_1\left(\frac{x+t}{2}\right) - \frac{1}{2} \int_{(x+t)/2}^{+\infty} q(u)\, R_1(u)\, du$$

$$- i \int_x^{+\infty} p(u)\, K(u, t-x+u)\, du + i \int_x^{(x+t)/2} p(u)\, K(u, t+x-u)\, du$$

$$+ \frac{1}{2} \int_x^{+\infty} q(u) \int_{t+x-u}^{t-x+u} K(u,\xi)\, d\xi du. \tag{2.9}$$

Similiarly, for $0 < x < a$, we obtain

$$R_2(x) - 1 = -\frac{i}{\alpha}\int_x^a p(t)\,R_2(t)\,dt - i\int_a^{+\infty} p(t)\,R_1(t)\,dt$$

$$R_3(x) - 1 = \frac{i}{\alpha}\int_x^a p(t)\,R_3(t)\,dt - i\int_a^{+\infty} p(t)\,R_1(t)\,dt,$$

where $K(x,t) \equiv 0$ for $|\xi| < \alpha t - \alpha a + a$. Further one can get the following integral equations for the kernel $K(x,t)$,

If $0 < x < a$, $\alpha x - \alpha a + a < t < -\alpha x + \alpha a - a$;

$$\begin{aligned}
K(x,t) &= i\frac{\alpha^+}{2\alpha^2}p\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)R_2\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)\\
&-\frac{\alpha^+}{2\alpha}\int_{(t+\alpha x+\alpha a-a)/2\alpha}^{a} q(u)\,R_2(u)\,du\\
&+i\frac{\alpha^-}{2\alpha^2}p\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)R_3\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)\\
&-\frac{\alpha^-}{2\alpha}\int_{(-t+\alpha x+\alpha a+a)/2\alpha}^{a} q(u)\,R_3(u)\,du - \frac{\alpha^+}{2}\int_a^{+\infty} q(u)\,R_1(u)\,du\\
&+i\frac{\alpha^-}{2}p\left(\frac{t-\alpha x+\alpha a+a}{2}\right)R_1\left(\frac{t-\alpha x+\alpha a+a}{2}\right)\\
&+\frac{\alpha^-}{2}\int_a^{(t-\alpha x+\alpha a+a)/2} q(u)\,R_1(u)\,du - \frac{i}{\alpha}\int_x^a p(u)\,K(u,t-\alpha x+\alpha u)\,du\\
&+\frac{i}{\alpha}\int_x^{(t+\alpha x+\alpha a-a)/2\alpha} p(u)\,K(u,t+\alpha x-\alpha u)\,du - \frac{1}{2\alpha}\int_x^a q(u)\int_{t+\alpha x-\alpha u}^{t-\alpha x+\alpha u} K(u,\xi)\,d\xi du\\
&-i\alpha^+\int_a^{+\infty} p(u)\,K(u,t-\alpha x+\alpha a-a+u)\,du - \frac{\alpha^+}{2}\int_a^{+\infty} q(u)\int_{t+\alpha x-\alpha a+a-u}^{t-\alpha x+\alpha a-a+u} K(u,\xi)\,d\xi du\\
&+i\alpha^-\int_a^{(t-\alpha x+\alpha a+a)/2} p(u)\,K(u,t-\alpha x+\alpha a+a-u)\,du\\
&+\frac{\alpha^-}{2}\int_a^{(t-\alpha x+\alpha a+a)/2} q(u)\int_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u} K(u,\xi)\,d\xi du
\end{aligned} \tag{2.10}$$

if $0 < x < a$, $-\alpha x + \alpha a - a < t < -\alpha x + \alpha a + a$;

$$\begin{aligned}
K(x,t) &= i\frac{\alpha^+}{2\alpha^2}p\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)R_2\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)\\
&-\frac{\alpha^+}{2\alpha}\int_{(t+\alpha x+\alpha a-a)/2\alpha}^{a} q(u)\,R_2(u)\,du
\end{aligned}$$

$$+i\frac{\alpha^-}{2\alpha^2}p\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)R_3\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)$$

$$-\frac{\alpha^-}{2\alpha}\int\limits_{(-t+\alpha x+\alpha a+a)/2\alpha}^{a}q(u)R_3(u)\,du-\frac{\alpha^+}{2}\int\limits_a^{+\infty}q(u)R_1(u)\,du$$

$$+i\frac{\alpha^-}{2}p\left(\frac{t-\alpha x+\alpha a+a}{2}\right)R_1\left(\frac{t-\alpha x+\alpha a+a}{2}\right)$$

$$+\frac{\alpha^-}{2}\int\limits_a^{(t-\alpha x+\alpha a+a)/2}q(u)R_1(u)\,du-\frac{i}{\alpha}\int\limits_x^a p(u)K(u,t-\alpha x+\alpha u)\,du$$

$$+\frac{i}{\alpha}\int\limits_x^{(t+\alpha x+\alpha a-a)/2\alpha}p(u)K(u,t+\alpha x-\alpha u)\,du$$

$$-\frac{1}{2\alpha}\int\limits_x^a q(t)\int\limits_{t+\alpha x-\alpha u}^{t-\alpha x+\alpha u}K(u,\xi)\,d\xi du$$

$$-i\alpha^+\int\limits_a^{+\infty}p(u)K(u,t-\alpha x+\alpha a-a+u)\,du$$

$$-\frac{\alpha^+}{2}\int\limits_a^{+\infty}q(u)\int\limits_{t+\alpha x-\alpha a+a-u}^{t-\alpha x+\alpha a-a+u}K(u,\xi)\,d\xi du$$

$$+i\alpha^-\int\limits_a^{(t-\alpha x+\alpha a+a)/2}p(u)K(u,t-\alpha x+\alpha a+a-u)\,du$$

$$+\frac{\alpha^-}{2}\int\limits_a^{(t-\alpha x+\alpha a+a)/2}q(u)\int\limits_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u}K(u,\xi)\,d\xi du \tag{2.11}$$

if $0<x<a$, $-\alpha x+\alpha a+a<t<+\infty$;

$$K(x,t)=i\frac{\alpha^+}{2}p\left(\frac{t+\alpha x-\alpha a+a}{2}\right)R_1\left(\frac{t+\alpha x-\alpha a+a}{2}\right)$$

$$-\frac{\alpha^+}{2}\int\limits_{(t+\alpha x-\alpha a+a)/2}^{+\infty}q(u)R_1(u)\,du$$

$$+i\frac{\alpha^-}{2}p\left(\frac{t-\alpha x+\alpha a+a}{2}\right)R_1\left(\frac{t-\alpha x+\alpha a+a}{2}\right)$$

$$-\frac{i}{\alpha}\int\limits_x^a p(u)K(u,t-\alpha x+\alpha u)\,du$$

$$+\frac{i}{\alpha}\int\limits_x^a p(u)K(u,t+\alpha x-\alpha u)\,du$$

$$-\frac{1}{2\alpha}\int\limits_x^a q(u)\int\limits_{t+\alpha x-\alpha u}^{t-\alpha x+\alpha u}K(u,\xi)\,d\xi du$$

$$-i\alpha^+\int\limits_a^{+\infty}p(u)K(u,t-\alpha x+\alpha a-a+u)\,du$$

$$+i\alpha^{+}\int_{a}^{(t+\alpha x-\alpha a+a)/2} p(u)\,K(u,t+\alpha x-\alpha a+a-u)\,du$$

$$-\frac{\alpha^{+}}{2}\int_{a}^{+\infty} q(u)\int_{t+\alpha x-\alpha a+a-u}^{t-\alpha x+\alpha a-a+u} K(u,\xi)\,d\xi du$$

$$-i\alpha^{-}\int_{a}^{+\infty} p(u)\,K(u,t+\alpha x-\alpha a-a+u)\,du$$

$$+i\alpha^{-}\int_{a}^{(t-\alpha x+\alpha a+a)/2} p(u)\,K(u,t-\alpha x+\alpha a+a-u)\,du$$

$$\frac{\alpha^{-}}{2}\int_{a}^{(t-\alpha x+\alpha a+a)/2} q(u)\int_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u} K(u,\xi)\,d\xi du. \tag{2.12}$$

Let's shown by the method of successive approximations that, equations of (2.10), (2.11) and (2.12) have solution $K(x,.)\in L_1(\mu^{+}(x),\infty)$ satisfying property of (2.2).

In the case of $x>a$

$$K_0(x,t)=\frac{i}{2}p\left(\frac{x+t}{2}\right)R_1\left(\frac{x+t}{2}\right)-\frac{1}{2}\int_{(x+t)/2}^{+\infty} q(u)\,R_1(u)\,du$$

$$K_1(x,t)=-i\int_{x}^{+\infty} p(u)\,K_0(u,t-x+u)\,du$$

$$+i\int_{x}^{(x+t)/2} p(u)\,K_0(u,t+x-u)\,du+\frac{1}{2}\int_{x}^{+\infty} q(u)\int_{t+x-u}^{t-x+u} K_0(u,\xi)\,d\xi du$$

$$K_n(x,t)=-i\int_{x}^{+\infty} p(u)\,K_{n-1}(u,t-x+u)\,du$$

$$+i\int_{x}^{(x+t)/2} p(u)\,K_{n-1}(u,t+x-u)\,du+\frac{1}{2}\int_{x}^{+\infty} q(u)\int_{t+x-u}^{t-x+u} K_{n-1}(u,\xi)\,d\xi du$$

$$K(x,t)=\sum_{n=0}^{\infty}K_n(x,t) \tag{2.13}$$

Let us show by induction that

$$\int_{\alpha x-\alpha a+a}^{+\infty}|K_n(x,t)|\,dt\leq\frac{\sigma^{n+1}(x)}{(n+1)!},$$

where,

$$\int_{x}^{+\infty}\left[2\,|p(t)|+(1+|t|)\,|q(t)|\right]dt=\sigma(x). \tag{2.14}$$

Indeed,

$$\int_x^{+\infty} |K_0(x,t)|\,dt \le \frac{1}{2}\int_x^{+\infty} \left|p\left(\frac{x+t}{2}\right)\right| \left|R_1\left(\frac{x+t}{2}\right)\right| dt$$

$$+\frac{1}{2}\int_x^{+\infty}\int_{(x+t)/2}^{+\infty} |q(u)|\,|R_1(u)|\,dudt.$$

We obtain

$$\int_x^{+\infty} |K_0(x,t)|\,dt \le \int_x^{+\infty} [2\,|p(t)| + (1+|t|)\,|q(t)|]\,dt.$$

Thus

$$\int_x^{+\infty} |K_0(x,t)|\,dt \le \sigma(x)\,. \tag{2.15}$$

For $n=1$

$$\int_x^{+\infty} |K_1(x,t)|\,dt \le \int_x^{+\infty}\int_x^{+\infty} |p(u)|\,|K_0(u,t-x+u)|\,dudt$$

$$+\int_x^{+\infty}\int_x^{(x+t)/2} |p(u)|\,|K_0(u,t+x-u)|\,dudt + \frac{1}{2}\int_x^{+\infty}\int_x^{+\infty} |q(u)| \int_{t+x-u}^{t-x+u} |K_0(u,\xi)|\,d\xi dudt.$$

$$\le \int_x^{+\infty} [2\,|p(t)| + (1+|t|)\,|q(t)|]\,\sigma(t)\,dt.$$

We get

$$\int_x^{+\infty} |K_1(x,t)|\,dt \le \frac{\sigma^2(x)}{2!}. \tag{2.16}$$

Suppose that

$$\int_x^{+\infty} |K_{n-1}(x,t)|\,dt \le \frac{\sigma^n(x)}{n!} \tag{2.17}$$

is valid for $n-1$. Let's show that

$$\int_x^{+\infty} |K_n(x,t)|\,dt \le \frac{\sigma^{n+1}(x)}{(n+1)!}. \tag{2.18}$$

$$\int_x^{+\infty} |K_n(x,t)|\,dt \le \int_x^{+\infty}\int_x^{+\infty} |p(u)|\,|K_{n-1}(u,t-x+u)|\,dudt$$

$$+\int_x^{+\infty}\int_x^{(x+t)/2} |p(u)|\,|K_{n-1}(u,t+x-u)|\,dudt + \frac{1}{2}\int_x^{+\infty}\int_x^{+\infty} |q(u)| \int_{t+x-u}^{t-x+u} |K_{n-1}(u,\xi)|\,d\xi dudt$$

$$\le \int_x^{+\infty} [2\,|p(t)| + (1+|t|)\,|q(t)|]\,\frac{\sigma^n(t)}{n!}dt.$$

Hence

$$\int_x^{+\infty} |K_n(x,t)|\, dt \le \frac{\sigma^{n+1}(x)}{(n+1)!}$$

is valid. It follows by virtue of last equation in the case of $x > a$, series, (2.13) convergents in $L_1(x,+\infty)$ for each fixed x, and then

$$\int_x^{+\infty} |K(x,t)|\, dt \le \sum_{n=0}^{\infty} \int_x^{+\infty} |K_n(x,t)|\, dt$$

$$\le 1 - 1 + \sigma(x) + \frac{\sigma^2(x)}{2!} + \ldots + \frac{\sigma^{n+1}(x)}{(n+1)!} + \ldots$$

$$= e^{\sigma(x)} - 1. \tag{2.19}$$

Now, we consider the case of $0 < x < a$.

Then, put $K(x,t) \equiv 0$ for $t > -\alpha x + \alpha a - a$ in the case of $\alpha x - \alpha a + a < t < -\alpha x + \alpha a - a$

$$\int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} |K_0(x,t)|\, dt \le \frac{\alpha^+}{2\alpha^2} \int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left| p\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)\right| \left| R_2\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)\right| dt$$

$$+\frac{\alpha^+}{2\alpha} \int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left(\int_{(t+\alpha x+\alpha a-a)/2\alpha}^{a} |q(u)|\, |R_2(u)|\, du \right) dt$$

$$+\frac{\alpha^-}{2\alpha^2} \int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left| p\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)\right| \left| R_3\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)\right| dt$$

$$+\frac{\alpha^-}{2\alpha} \int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left(\int_{(-t+\alpha x+\alpha a+a)/2\alpha}^{a} |q(u)|\, |R_3(u)|\, du \right) dt$$

$$+\frac{\alpha^+}{2} \int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left(\int_a^{+\infty} |q(u)|\, |R_1(u)|\, du \right) dt$$

$$+\frac{\alpha^-}{2} \int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left| p\left(\frac{t-\alpha x+\alpha a+a}{2}\right)\right| \left| R_1\left(\frac{t-\alpha x+\alpha a+a}{2}\right)\right| dt$$

$$+\frac{\alpha^-}{2} \int_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left(\int_a^{(t-\alpha x+\alpha a+a)/2} |q(u)|\, |R_1(u)|\, du \right) dt$$

$$\le \frac{\alpha^+}{\alpha} \int_x^{+\infty} |p(t)|\, dt + \alpha^+ \int_x^{+\infty} (1+|t|)\, |q(t)|\, dt - \frac{\alpha^-}{\alpha} \int_x^{+\infty} |p(t)|\, dt$$

$$+\alpha^- \int_x^{+\infty} (1+|t|)\, |q(t)|\, dt + \alpha\alpha^+ \int_x^{+\infty} (1+|t|)\, |q(t)|\, dt$$

$$+\alpha^- \int_x^{+\infty} |p(t)|\, dt + 2\alpha\alpha^- \int_x^{+\infty} (1+|t|)\, |q(t)|\, dt$$

$$< \alpha^{+} \int\limits_{x}^{+\infty} \left[2\left|p\left(t\right)\right| + \left(1+|t|\right)\left|q\left(t\right)\right|\right] dt$$

$$+\alpha^{-} \int\limits_{x}^{+\infty} \left[2\left|p\left(t\right)\right| + \left(1+|t|\right)\left|q\left(t\right)\right|\right] dt$$

$$+\alpha\left(\alpha^{+}+2\alpha^{-}\right) \int\limits_{x}^{+\infty} \left(1+|t|\right)\left|q\left(t\right)\right| dt + \alpha^{-} \int\limits_{x}^{+\infty} \left|p\left(t\right)\right| dt$$

$$< \alpha^{+}\sigma\left(x\right) + \alpha^{-}\sigma\left(x\right) + 2\alpha \int\limits_{x}^{+\infty} \left(1+|t|\right)\left|q\left(t\right)\right| dt + 4\alpha \int\limits_{x}^{+\infty} \left|p\left(t\right)\right| dt$$

$$< \left(\alpha^{+}+\alpha^{-}+2\alpha\right)\sigma\left(x\right) = c_{1}\sigma\left(x\right). \tag{2.20}$$

Hence

$$\int\limits_{\alpha x-\alpha a+a}^{-\alpha x+\alpha a-a} \left|K_{0}\left(x,t\right)\right| dt \leq c_{1}\sigma\left(x\right).$$

Put $K\left(x,t\right) \equiv 0$ for $t < -\alpha x+\alpha a-a$ in the case of $-\alpha x+\alpha a-a < t < -\alpha x+\alpha a+a$ to get

$$\int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left|K_{0}\left(x,t\right)\right| dt \leq \frac{\alpha^{+}}{2\alpha^{2}} \int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left|p\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)\right| \left|R_{2}\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)\right| dt$$

$$+\frac{\alpha^{+}}{2\alpha} \int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left(\int\limits_{(t+\alpha x+\alpha a-a)/2\alpha}^{a} \left|q\left(u\right)\right| \left|R_{2}\left(u\right)\right| du\right) dt$$

$$+\frac{\alpha^{-}}{2\alpha^{2}} \int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left|p\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)\right| \left|R_{3}\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)\right| dt$$

$$+\frac{\alpha^{-}}{2\alpha} \int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left(\int\limits_{(-t+\alpha x+\alpha a+a)/2\alpha}^{a} \left|q\left(u\right)\right| \left|R_{3}\left(u\right)\right| du\right) dt$$

$$+\frac{\alpha^{+}}{2} \int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left(\int\limits_{a}^{+\infty} \left|q\left(u\right)\right| \left|R_{1}\left(u\right)\right| du\right) dt$$

$$+\frac{\alpha^{-}}{2} \int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left|p\left(\frac{t-\alpha x+\alpha a+a}{2}\right)\right| \left|R_{1}\left(\frac{t-\alpha x+\alpha a+a}{2}\right)\right| dt$$

$$+\frac{\alpha^{-}}{2} \int\limits_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a} \left(\int\limits_{a}^{(t-\alpha x+\alpha a+a)/2} \left|q\left(u\right)\right| \left|R_{1}\left(u\right)\right| du\right) dt$$

$$< \frac{\alpha^{+}}{\alpha} \int\limits_{x}^{+\infty} \left|p\left(t\right)\right| dt + \alpha^{+} \int\limits_{x}^{+\infty} \left(1+|t|\right)\left|q\left(t\right)\right| dt$$

$$+\frac{\alpha^{-}}{\alpha} \int\limits_{x}^{+\infty} \left|p\left(t\right)\right| dt + \alpha^{-} \int\limits_{x}^{+\infty} \left(1+|t|\right)\left|q\left(t\right)\right| dt$$

$$+\alpha^{+} \int\limits_{x}^{+\infty} \left(1+|t|\right)\left|q\left(t\right)\right| dt + \alpha^{-} \int\limits_{x}^{+\infty} \left|p\left(t\right)\right| dt$$

$$+\alpha\alpha^{-}\int_{x}^{+\infty}(1+|t|)\,|q(t)|\,dt$$

$$<\alpha^{+}\int_{x}^{+\infty}[2\,|p(t)|+(1+|t|)\,|q(t)|]\,dt$$

$$+\alpha^{-}\int_{x}^{+\infty}[2\,|p(t)|+(1+|t|)\,|q(t)|]\,dt$$

$$+\left(\alpha\alpha^{-}+\alpha^{+}\right)\int_{x}^{+\infty}(1+|t|)\,|q(t)|\,dt+\alpha^{-}\int_{x}^{+\infty}|p(t)|\,dt$$

$$<\alpha^{+}\sigma(x)+\alpha^{-}\sigma(x)+\alpha\alpha^{+}\int_{x}^{+\infty}[2\,|p(t)|+(1+|t|)\,|q(t)|]\,dt$$

$$=\left(\alpha^{+}+\alpha^{-}+\alpha\alpha^{+}\right)\sigma(x)=c_{2}\sigma(x)\,. \tag{2.21}$$

Thus

$$\int_{-\alpha x+\alpha a-a}^{-\alpha x+\alpha a+a}|K_{0}(x,t)|\,dt\le c_{2}\sigma(x)$$

is obtained.

For $-\alpha x+\alpha a+a<t<+\infty$,

$$\int_{-\alpha x+\alpha a+a}^{+\infty}|K_{0}(x,t)|\,dt\le\frac{\alpha^{+}}{2}\int_{-\alpha x+\alpha a+a}^{+\infty}\left|p\left(\frac{t+\alpha x-\alpha a+a}{2}\right)\right|\left|R_{1}\left(\frac{t+\alpha x-\alpha a+a}{2}\right)\right|dt$$

$$+\frac{\alpha^{+}}{2}\int_{-\alpha x+\alpha a+a}^{+\infty}\left(\int_{(t+\alpha x-\alpha a+a)/2}^{+\infty}|q(u)|\,|R_{1}(u)|\,du\right)dt$$

$$+\frac{\alpha^{-}}{2}\int_{-\alpha x+\alpha a+a}^{+\infty}\left|p\left(\frac{t-\alpha x+\alpha a+a}{2}\right)\right|\left|R_{1}\left(\frac{t-\alpha x+\alpha a+a}{2}\right)\right|dt$$

$$<\alpha^{+}\int_{x}^{+\infty}|p(t)|\,dt+\alpha^{+}\int_{x}^{+\infty}(1+|t|)\,|q(t)|\,dt+\alpha^{+}\int_{x}^{+\infty}|p(t)|\,dt$$

$$=\alpha^{+}\int_{x}^{+\infty}[2\,|p(t)|+(1+|t|)\,|q(t)|]\,dt=\alpha^{+}\sigma(x)\,. \tag{2.22}$$

So

$$\int_{-\alpha x+\alpha a+a}^{+\infty}|K_{0}(x,t)|\,dt\le\alpha^{+}\sigma(x)\,.$$

Then, it follows from $(2.20)\,,(2.21)$ and (2.22) for $0<x<a$

$$\int_{\alpha x-\alpha a+a}^{+\infty}|K_{0}(x,t)|\,dt\le C\sigma(x)\,, \tag{2.23}$$

where $C=c_{1}+c_{2}+\alpha^{+}$, $c_{1}=\alpha^{+}+\alpha^{-}+2\alpha$, $c_{2}=\alpha^{+}+\alpha^{-}+\alpha\alpha^{+}$.

Now, for $n=1$, if one evaluate integral $\int\limits_{\alpha x-\alpha a+a}^{+\infty}|K_1(x,t)|\,dt$ then

$$\int\limits_{\alpha x-\alpha a+a}^{+\infty}|K_1(x,t)|\,dt\leq\frac{1}{\alpha}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_x^a|p(u)|\,|K_0(u,t-\alpha x+\alpha u)|\,du\right)dt$$
$$+\frac{1}{\alpha}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_x^a|p(u)|\,|K_0(u,t+\alpha x-\alpha u)|\,du\right)dt$$
$$+\frac{1}{2\alpha}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_x^a|q(u)|\int\limits_{t+\alpha x-\alpha u}^{t-\alpha x+\alpha u}|K_0(u,\xi)|\,d\xi du\right)dt$$
$$+\alpha^+\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{+\infty}|p(u)|\,|K_0(u,t-\alpha x+\alpha a-a+u)|\,du\right)dt$$
$$+\alpha^+\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{(t+\alpha x-\alpha a+a)/2}|p(u)|\,|K_0(u,t+\alpha x-\alpha a+a-u)|\,du\right)dt$$
$$+\frac{\alpha^+}{2}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{+\infty}|q(u)|\int\limits_{t+\alpha x-\alpha a+a-u}^{t-\alpha x+\alpha a-a+u}|K_0(u,\xi)|\,d\xi du\right)dt$$
$$+\alpha^-\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{+\infty}|p(u)|\,|K_0(u,t+\alpha x-\alpha a-a+u)|\,du\right)dt$$
$$+\alpha^-\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{(t-\alpha x+\alpha a+a)/2}|p(u)|\,|K_0(u,t-\alpha x+\alpha a+a-u)|\,du\right)dt$$
$$+\frac{\alpha^-}{2}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{(t-\alpha x+\alpha a+a)/2}|q(u)|\int\limits_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u}|K_0(u,\xi)|\,d\xi du\right)dt$$
$$<C\int\limits_x^a\left[2|p(t)|+(1+|t|)\,|q(t)|\right]\sigma(t)\,dt$$
$$+C\int\limits_a^{+\infty}\left[2|p(t)|+(1+|t|)\,|q(t)|\right]\sigma(t)\,dt. \qquad (2.24)$$

It follows from (2.24)

$$\int\limits_{\alpha x-\alpha a+a}^{+\infty}|K_1(x,t)|\,dt\leq C\frac{\sigma^2(x)}{2!}. \qquad (2.25)$$

Assume that

$$\int\limits_{\alpha x-\alpha a+a}^{+\infty}|K_{n-1}(x,t)|\,dt\leq C\frac{\sigma^n(x)}{n!} \qquad (2.26)$$

is valid for $n-1$.Let's show that

$$\int\limits_{\alpha x-\alpha a+a}^{+\infty}|K_n(x,t)|\,dt\leq C\frac{\sigma^{n+1}(x)}{(n+1)!} \qquad (2.27)$$

is valid for n.

$$\int\limits_{\alpha x-\alpha a+a}^{+\infty}|K_n(x,t)|\,dt \le \frac{1}{\alpha}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_x^a |p(u)|\,|K_{n-1}(u,t-\alpha x+\alpha u)|\,du\right)dt$$

$$+\frac{1}{\alpha}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_x^a |p(u)|\,|K_{n-1}(u,t+\alpha x-\alpha u)|\,du\right)dt$$

$$+\frac{1}{2\alpha}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_x^a |q(u)|\int\limits_{t+\alpha x-\alpha u}^{t-\alpha x+\alpha u}|K_{n-1}(u,\xi)|\,d\xi du\right)dt$$

$$+\alpha^{+}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{+\infty}|p(u)|\,|K_{n-1}(u,t-\alpha x+\alpha a-a+u)|\,du\right)dt$$

$$+\alpha^{+}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{(t+\alpha x-\alpha a+a)/2}|p(u)|\,|K_{n-1}(u,t+\alpha x-\alpha a+a-u)|\,du\right)dt$$

$$+\frac{\alpha^{+}}{2}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{+\infty}|q(u)|\int\limits_{t+\alpha x-\alpha a+a-u}^{t-\alpha x+\alpha a-a+u}|K_{n-1}(u,\xi)|\,d\xi du\right)dt$$

$$+\alpha^{-}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{+\infty}|p(u)|\,|K_{n-1}(u,t+\alpha x-\alpha a-a+u)|\,du\right)dt$$

$$+\alpha^{-}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{(t-\alpha x+\alpha a+a)/2}|p(u)|\,|K_{n-1}(u,t-\alpha x+\alpha a+a-u)|\,du\right)dt$$

$$+\frac{\alpha^{-}}{2}\int\limits_{\alpha x-\alpha a+a}^{+\infty}\left(\int\limits_a^{(t-\alpha x+\alpha a+a)/2}|q(u)|\int\limits_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u}|K_{n-1}(u,\xi)|\,d\xi du\right)dt$$

$$<\frac{C}{n!}\int\limits_x^a[2|p(t)|+(1+|t|)|q(t)|]\,\sigma^n(t)\,dt$$

$$+\frac{C}{n!}\int\limits_a^{+\infty}[2|p(t)|+(1+|t|)|q(t)|]\,\sigma^n(t)\,dt.$$

Thus

$$\int\limits_{\alpha x-\alpha a+a}^{+\infty}|K_n(x,t)|\,dt \le C\frac{\sigma^{n+1}(x)}{(n+1)!} \tag{2.28}$$

Remark: For simplicity, α was taken as $\alpha > 1$. Other case can be treated in the same way.

Finally, let investigate correlation between the kernel function $K(x,t)$ and coefficients $p(x), q(x)$ and $\rho(x)$ in the equation

$$-y'' + [2\lambda p(x) + q(x)]\,y = \lambda^2\rho(x)\,y.$$

For $0 < x < a$,

$$e(x,\lambda) = \alpha^{+}R_2(x)\,e^{i\lambda\mu^{+}(x)} + \alpha^{-}R_3(x)\,e^{i\lambda\mu^{-}(x)} + \int\limits_{\mu^{+}(x)}^{+\infty}K(x,t)\,e^{i\lambda t}dt.$$

with $\mu^+(x) < \mu^-(x) < +\infty$. If we put $e(x,\lambda)$ and $e''(x,\lambda)$ instead of y ve y'' in (1.1), we get;

$$\alpha^+ R_2''(x) + 2i\lambda\alpha\alpha^+ R_2'(x) - 2\lambda\alpha^+ p(x) R_2(x) + 2ip(x) K(x,\mu^+(x))$$

$$-\alpha^+ q(x) R_2(x) - 2\alpha\frac{dK(x,\mu^+(x))}{dx} = 0$$

$$\alpha^- R_3''(x) - 2i\lambda\alpha\alpha^- R_3'(x) - 2\lambda\alpha^- p(x) R_3(x) - 2ip(x) K(x,\mu^-(x)-0)$$

$$+2ip(x) K(x,\mu^-(x)+0) - \alpha^- q(x) R_3(x)$$

$$-2\alpha\frac{dK(x,\mu^-(x)-0)}{dx} + 2\alpha\frac{dK(x,\mu^-(x)+0)}{dx} = 0$$

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \alpha^2\frac{\partial^2 K(x,t)}{\partial t^2} + 2ip(x)\frac{\partial K(x,t)}{\partial t} = q(x) K(x,t)$$

$$\lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial x} = 0, \quad \lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial t} = 0$$

In the case $x > a$, since $\mu^+(x) = x$ and $\alpha = 1$, then the same properties of the kernel function.can be easily obtained.

Hence, the proof is completed.

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