On the solutions of the quadratic pencil of the Sturm-Liouville equation with steplike potential

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Received: 12.02.2008, Accepted: 18.02.2008

Abstract: The Jost solutions of the quadratic pencil of the Sturm-Liouville equation with steplike potential are investigated on the real line. The Jost solutions are defined when the potential is asymptotic to different constants as \( x \to +\infty, -\infty \). The integral representations are obtained for the Jost solutions and some spectral properties are investigated.

Keywords: Sturm-Liouville equation with steplike potentials, energy-dependent Sturm-Liouville equation, transformation operators, the Jost solution, spectral analysis of differential operators

Basamak türünden potansiyele sahip Sturm-Liouville denkleminin ikinci dereceden demetinin çözümleri üzerine

Özet: Bu çalışmada reel eksende basamak türünden potansiyele sahip Sturm-Liouville denkleminin ikinci dereceden demetinin Jost çözümleri incelenmektedir. Potansiyelin \( x \to +\infty, -\infty \) iken farklı sabitlere yaklaşması durumu için Jost çözümleri belirlenmektedir. Jost çözümleri için integral gösterilimler elde edilmekte ve bazı spektral özellikleri incelenmektedir.

Anahtar kelimeler: Basamak türden potansiyele sahip Sturm-Liouville denklemi, enerji bağlı Sturm-Liouville denklemi, dönüşüm operatörleri, Jost çözümü, diferansiyel operatörlerin spektral analizi
1. Introduction

This article deals with the investigation of the Jost solutions of the quadratic pencil of the Sturm-Liouville equation when the potential is asymptotic to different constants as \( x \to \pm \infty \). It is well known that the Jost solutions play an important role in the investigation direct and inverse scattering problems for Sturm-Liouville operators. The direct and inverse problems for the classical Sturm-Liouville operators with steplike potentials was first studied by Buslayev and Fomin [1]. After Buslayev and Fomin's work Hruslov [2] applied their results to the solution of the Cauchy problem for the Korteweg-de Vries equation with steplike initial profile. Later A.Cohen [5] constructed a counterexample asserted some incorrects in [1]. The direct and inverse scattering for steplike potentials for the Sturm-Liouville equation was developed and completely solved in [6,7].

The full-line direct and inverse scattering problems for the quadratic pencil of the Sturm-Liouville equation with potentials having zero limit at infinity, as a generalization of the Marchenko method, was investigated in [3,4].

It is natural to set and solve the analogous problems for the quadratic pencil of the Sturm-Liouville equation with steplike potentials. But in this case there are some difficulties with the construction of the Jost solutions which play a basic role in solving the direct and inverse scattering problems. The aim of this paper is to obtain the integral representation for the Jost solutions.

2. The Jost Solutions

On the entire real line let us consider the generalized Sturm-Liouville equation

\[
-y'' + V(x, \lambda) y = \lambda^2 y, \quad -\infty < x < +\infty,
\]

(1)

with the energy dependent potential \( V(x, \lambda) = q(x) + 2\lambda p(x) \), where \( q(x), p(x) \) are real valued functions defined on the entire real line and \( \lambda \) is a complex parameter. Assume that the following conditions are satisfied:

(i) the function \( q(x) \) is locally summable and has different limit values:

\[
\lim_{x \to +\infty} q(x) = c_1^2, \quad \lim_{x \to -\infty} q(x) = c_2^2,
\]

where \( 0 \leq c_1^2 \leq c_2^2 \);
(ii) the function \( p(x) \) is absolutely continuous on each finite segment \([\alpha;\beta] \subset (-\infty;+\infty)\) and
\[
\int_{-\infty}^{\infty} (|x|+1)[|q(x)-c_j|^2 + 2c_j|p(x)| + |p'(x)||dx < \infty, \quad j = 1;2;
\]

(iii) Denote by
\[
\sigma_j(x) = \pm \int_{x}^{\pm\infty} [\frac{1}{|q(t) - c_j|^2} - 2c_j|p(t)| + |p'(t)||dt,
\]
\[
\tau_j(x) = \pm \int_{x}^{\pm\infty} [\frac{1}{|q(t) - c_j|^2} + (2c_j + 1)|p(t)| + |p'(t)||dt,
\]
where \( j = 1;2 \).

Let \( \Gamma_j \) is the of Riemann surface of the function \( k_j(\lambda) = \sqrt{\lambda^2 - c_j^2}, \Gamma_j^+ \) is the upper sheet of this surface and the "boundary" of \( \Gamma_j^+ \) is \( \partial \Gamma_j^+ \) which is the set of points of the upper and the lower lips of the \( \lambda \) -plane cut along the rays \(|\lambda| > |c_j| \) \((j = 1;2)\).

For \( \lambda \in \partial \Gamma_j^+ \) we define the Jost solutions \( f_j(x,\lambda) \) of the equation (1) with the asymptotic conditions
\[
\lim_{\lambda \to +\infty} f_1(x,\lambda) e^{-ik_j(x)} = 1,
\]
\[
\lim_{\lambda \to -\infty} f_2(x,\lambda) e^{-ik_j(x)} = 1.
\]

Rewrite the equation (1) in the form
\[
-y'' + (q(x) - c_j^2)y + 2\sqrt{k_j^2 + c_j^2} p(x)y = k_j^2 y,
\]
where \( k_j = k_j(\lambda) \). It is easy to see that the Jost solution \( f_j(x,\lambda) \) satisfies the integral equation
\[
f_j(x,\lambda) = e^{\pm ik_j x} + \int_{x}^{\pm\infty} \frac{\sin k_j(t-x)}{k_j} (q(t) - c_j^2 + 2k_j p(t) + 2(\sqrt{k_j^2 + c_j^2} - k_j) p(t)) f_j(t,\lambda) dt,
\]
where "+" and "-" corresponds to \( j = 1 \) and \( j = 2 \) consequently.

**Theorem 1:** If the conditions (i)-(ii) are satisfied then for all \( \lambda \in \Gamma_j^+ \cup \partial \Gamma_j^+ \) the solution \( f_j(x,\lambda) \) can be represented in the form
\[ f_j(x, \lambda) = R_j(x)e^{ \frac{z_ik_j(\lambda)x}{2} } \pm \int_x^{\infty} A_j(x,t)e^{\frac{z_ik_j(\lambda)t}{2}} dt, \quad (5) \]

where
\[
R_j(x) = \exp \left( \pm i \int_x^{\infty} p(t) dt \right), \quad (6)
\]
and the kernel \( A_j(x,t) \) satisfies the inequality
\[
|A_j(x,t)| \leq \frac{1}{2} \sigma_j \left( \frac{x + t}{2} \right) \exp(\tau_j(x)). \quad (7)
\]
In addition,
\[
A_j(x,x) = \pm \frac{1}{2} \int_x^{\infty} (q(t) - c_j^2) R_j(t) dt + \frac{1}{2i} p(x) R_j(x) \pm \int_x^{\infty} A_j(t,t) p(t) dt. \quad (8)
\]

**Proof:** Suppose that the integral equation (4) has the solution of the form (5). Substituting the expression (5) of \( f_j(x, \lambda) \) in (4) we have
\[
(R_j(x) - 1)e^{\frac{z_ik_jx}{2} } \pm \int_x^{\infty} A_j(x,t)e^{\frac{z_ik_jt}{2} } dt = 
\int_x^{\infty} \sin k_j(t-x) \frac{q(t) - c_j^2}{k_j} [q(t) - c_j^2 + 2k_j p(t) + 2(\sqrt{k_j^2 + c_j^2} - k_j) p(t)] \times 
\left[ R_j(t)e^{\frac{z_ik_jt}{2} } \pm \int_x^{\infty} A_j(t,\xi)e^{\frac{z_ik_j\xi}{2}} d\xi \right] dt. \quad (9)
\]
Rewrite (9) in the form
\[
(R_j(x) - 1)e^{\frac{z_ik_jx}{2} } \pm \int_x^{\infty} A_j(x,t)e^{\frac{z_ik_jt}{2} } dt = 
\int_x^{\infty} \left( e^{\frac{z_ik_j(2\tau-x)}{2} } \pm e^{\frac{z_ik_j(x-2\tau)}{2} } \right) (q(t) - c_j^2) R_j(t) dt \mp 

i \int_x^{\infty} \left( e^{\frac{z_ik_j(2\tau-x)}{2} } \pm e^{\frac{z_ik_j(x-2\tau)}{2} } \right) p(t) R_j(t) dt \pm 

\int_x^{\infty} \left( e^{\frac{z_ik_j(\xi+t-x)}{2} } \pm e^{\frac{z_ik_j(\xi-t-x)}{2} } \right) A_j(t,\xi) d\xi - 

\int_x^{\infty} p(t) dt \int_x^{\infty} \left( e^{\frac{z_ik_j(\xi-t-x)}{2} } \pm e^{\frac{z_ik_j(\xi+t-x)}{2} } \right) A_j(t,\xi) d\xi + 

\int_x^{\infty} p(t) dt \int_x^{\infty} \left( e^{\frac{z_ik_j(\xi-t-x)}{2} } \pm e^{\frac{z_ik_j(\xi+t-x)}{2} } \right) A_j(t,\xi) d\xi \quad (9')
\]
Now let us require \( R_j(x) \) to satisfy
From here we have

\[ R_j(x) = \exp \left( \mp i \int_{x}^{+\infty} p(t)dt \right) \]

Further, using the formulas (see [8])

\[ \frac{e^{\pm ik_j(x-t)} - e^{\pm ik_j(x-t')}}{2ik_j} = \pm \int_{x-t}^{x-t'} e^{\pm ik_j s} ds, \]

\[ \frac{\sqrt{k_j^2 + c_j^2 - k_j}}{ik_j} = ic_j \int_{x}^{+\infty} \left[ \int_{0}^{t} I_i(c_j \xi) d\xi \right] d\xi, \quad \text{Im} k_j \geq 0, \]

where \( I_i(.) \) is the Bessel function, after some simple transformations we have the following equality:

\[ \pm \int_{x}^{+\infty} A_j(x,t) e^{\pm ik_j t} dt = \int_{x}^{+\infty} e^{\pm ik_j t} dt \left( \frac{1}{2} \right)^{\pm \infty} \int_{x-t}^{x-t'} (q(s) - c_j^2) R_j(s) ds \pm \]

\[ \frac{1}{2i} p\left( \frac{x+t}{2} \right) R_j\left( \frac{x+t}{2} \right) \mp ic_j \int_{x}^{t} \frac{I_i(c_j \xi)}{\xi} p(s) R_j(s) ds \pm \]

\[ \frac{1}{2} \int_{x-t}^{x-t'} (q(s) - c_j^2) ds \int_{x-s}^{x} A_j(s,\xi) d\xi + i \int_{x}^{+\infty} p(s) A_j(s,t+s-x) ds - \]

\[ i \int_{x}^{+\infty} p(s) A_j(s,t+s-x) ds \mp ic_j \int_{0}^{t} \frac{I_i(c_j \alpha)}{\alpha} p(s) ds \int_{x}^{+\infty} A_j(s,\xi) d\xi \left\{ \right. \]

(12)

where \( A_j(x,t) = 0 \) for \( \pm t < \pm x \). From (12) we get that if the function \( A_j(x,t) \)

(\( A_j(x,t) = 0 \) for \( \pm t < \pm x \)) satisfies the integral equation

\[ A_j(x,t) = \pm \frac{1}{2} \int_{x-t}^{x-t'} (q(s) - c_j^2) R_j(s) ds + \frac{1}{2i} p\left( \frac{x+t}{2} \right) R_j\left( \frac{x+t}{2} \right) - \]

\[ ic_j \int_{0}^{t} \frac{I_i(c_j \xi)}{\xi} p(s) R_j(s) ds + \int_{x-t}^{x-t'} \frac{1}{2} (q(s) - c_j^2) ds \int_{x-s}^{x} A_j(s,\xi) d\xi \pm \]
then the function \( f_j(x, \lambda) \) expressed by (5) is the Jost solution of the equation (1) and conversely, if the function (5) is the Jost solution of the equation (1) then the kernel \( A_j(x, t) \) satisfies the integral equation (13). Let us set \( t - x = 2v, t + x = 2u \) and \( H_j(u, v) = A_j(u - v, u + v) \). Then the integral equation (13) takes the following form:

\[
H_j(u, v) = \pm \frac{1}{2} \int_0^{2\pi} \frac{L}{\alpha} \int_{\alpha - \xi}^{\alpha + \xi} p(s) R_j(s) ds + \frac{1}{2i} p(u) R_j(u) -
\]

\[
ic_j \int_0^{2\pi} \frac{L}{\xi} \int_{\alpha - \xi}^{\alpha + \xi} p(s) R_j(s) ds + \int_{\alpha - \xi}^{\alpha + \xi} d\alpha \int_0^{2\pi} (q(\alpha - \beta) - c_j^2) H_j(\alpha, \beta) d\beta \pm
\]

\[
i \int_0^{2\pi} p(\alpha - v) H_j(\alpha, v) d\alpha \mp i \int_0^{2\pi} p(u - \beta) H_j(u, \beta) d\beta \mp
\]

\[
2ic \int_0^{2\pi} \frac{L}{\xi} \int_{\alpha - \xi}^{\alpha + \xi} p(\alpha - \beta) H_j(\alpha, \beta) d\beta
\]

(14)

To solve the integral equation (14) we apply the method of successive approximations. Let us define

\[
H_j^{(0)}(u, v) = \pm \frac{1}{2} \int_0^{2\pi} (q(s) - c_j^2) R_j(s) ds + \frac{1}{2i} p(u) R_j(u) -
\]

\[
ic_j \int_0^{2\pi} \frac{L}{\xi} \int_{\alpha - \xi}^{\alpha + \xi} p(s) R_j(s) ds,
\]

\[
H_j^{(m)}(u, v) = \int_{\alpha - \xi}^{\alpha + \xi} d\alpha \int_0^{2\pi} (q(\alpha - \beta) - c_j^2) H_j^{(m-1)}(\alpha, \beta) d\beta \pm
\]

\[
i \int_0^{2\pi} p(\alpha - v) H_j^{(m-1)}(\alpha, v) d\alpha \mp i \int_0^{2\pi} p(u - \beta) H_j(u, \beta) d\beta \mp
\]

\[
2ic \int_0^{2\pi} \frac{L}{\xi} \int_{\alpha - \xi}^{\alpha + \xi} p(\alpha - \beta) H_j^{(m-1)}(\alpha, \beta) d\beta, \ m = 1, 2, ...
\]

It is easy to obtain the estimates

\[
|H_j^{(0)}(u, v)| \leq \frac{1}{2} \sigma_j(u), \quad |H_j^{(m)}(u, v)| \leq \frac{1}{2} \sigma_j(u) \frac{\tau_j(u - v)}{m!}
\]

(15)

From (15) we have that the series \( \sum_{m=0}^{\infty} H_j^{(m)}(u, v) \) converges absolutely and uniformly, the sum \( H_j(u, v) \) of the series is a unique solution of the integral equation (14) and
\[ |H_j(u,v)| \leq \frac{1}{2} \sigma_j(u) \exp(\tau_j(u-v)) \quad (16) \]

Since \( u = x + t, v = t - x \) and \( H_j(u,v) = A_j(x,t) \) the estimation (7) immediately is obtained from (16). The equality (8) is obtained from (13). The proof is completed. 

(8) implies that

\[
A_j(x,x) = \frac{1}{2} e^{-i \alpha_j(x)} \left\{ -i p(x) + \int_x^{2\pi} [q(t) - c_j^2 + p^2(t)] dt \right\} e^{i \alpha_j(x)}, \quad (17)
\]

where \( \alpha_j(x) = \pm \int_x^{2\pi} p(t) dt \). Setting \( B_j(x,t) = \text{Re} A_j(x,t), K_j(x,t) = \text{Im} A_j(x,t) \) and taking into account the expression (6) we have

\[
\alpha_j(x) = \pm 2 \int_x^{2\pi} \left[ B_j(t,t) \sin \alpha_j(t) - K_j(t,t) \cos \alpha_j(t) \right] dt, \quad (18)
\]

\[
q(x) - c_j^2 + p^2(x) = \pm 2 \frac{d}{dx} \left( B_j(x,x) \sin \alpha_j(x) - K_j(x,x) \cos \alpha_j(x) \right). \quad (19)
\]

From (5) it is also obtained that uniformly for \( \lambda \in \Gamma_j^+ \) the following asymptotic formulas are hold:

\[
f_j(x,\lambda) = e^{ \pm ik_j(\lambda) } [1 + o(1)], \quad x \to \pm \infty \]

\[
f_j'(x,\lambda) = e^{ \pm ik_j(\lambda) } [ \pm i k_j(\lambda) e^{i \alpha_j(x)} + o(1)], \quad x \to \pm \infty \quad (20)
\]

where "+" corresponds to \( j = 1 \) and "-" corresponds to \( j = 2 \).

Since \( p(x) \) and \( q(x) \) are real functions then for all \( \lambda \in \partial \Gamma_j, \lambda \neq \pm c_j \) the equation (1) has solutions \( f_j(x,\lambda) \) and \( \overline{f_j(x,\lambda)} \). Let \( W[y_1,y_2] = y'_1(x)y_2(x) - y_1(x)y'_2(x) \) is the Wronskian of the functions \( y_1(x) \) and \( y_2(x) \). From the asymptotic formulas (20) we have

\[
W[f_j(x,\lambda),\overline{f_j(x,\lambda)}] = (-1)^{j+1} 2ik_j. \quad (21)
\]

Therefore, for \( \lambda \in \partial \Gamma_j^+, \lambda \neq \pm c_j \) the functions \( f_j(x,\lambda) \) and \( \overline{f_j(x,\lambda)} \) are linearly independent solutions of the equation (1) and the following equalities take place:

\[
f_2(x,\lambda) = a_1(\lambda) f_1(x,\lambda) + b_1(\lambda) f_2(x,\lambda), \lambda \in \partial \Gamma_1^+, \lambda \neq \pm c_1, \quad (22)
\]

\[
f_1(x,\lambda) = a_2(\lambda) f_2(x,\lambda) + b_2(\lambda) f_1(x,\lambda), \lambda \in \partial \Gamma_2^+, \lambda \neq \pm c_2, \quad (23)
\]
Taking into account (21) we have
\[ a_j(\lambda) = \frac{1}{2ik_j} W[f_1, f_2], \lambda \in \partial \Gamma_j^+, \lambda \neq \pm|c_j|. \]  
(24)

\[ b_1(\lambda) = \frac{1}{2ik_1} W[f_2, \overline{f_1}], \lambda \in \partial \Gamma_1^+, \lambda \neq \pm|k_1|. \]

\[ b_2(\lambda) = \frac{1}{2ik_2} W[f_2, f_1], \lambda \in \partial \Gamma_2^+, \lambda \neq \pm|c_2|. \]  
(25)

From (24) and (25) we obtain
\[ \frac{k_2(\lambda)}{k_1(\lambda)} |k_j(\lambda)|^2 = 1 + \frac{k_2(\lambda)}{k_1(\lambda)} |b_j(\lambda)|^2, |\lambda| > |c_1|. \]

\[ \frac{k_1(\lambda)}{k_2(\lambda)} |k_j(\lambda)|^2 = 1 + \frac{k_1(\lambda)}{k_2(\lambda)} |b_j(\lambda)|^2, |\lambda| > |c_2|. \]  
(26)

We also obtain that
\[ k_j(\lambda)a_j(\lambda) = k_2(\lambda)a_2(\lambda), k_j(\lambda)b_j(\lambda) = k_2(-\lambda)\overline{b_2(\lambda)}, |\lambda| > |c_j|. \]

(27)

Moreover, since \( f_j(x, \lambda) \) can be analytically continued into the upper sheet \( \Gamma_j^+ \) then from (24), (25) we conclude that the function \( a_j(\lambda) \) also can be analytically continued into \( \Gamma_j^+ \):
\[ a_j(\lambda) = \frac{1}{2ik_j} W[f_1, f_2], \lambda \in \Gamma_j^+ \]  
(28)

It is clear that for \( \lambda \) which is not real or for \( \lambda \in (-|c_2|, |c_2|) \) the equation (1) has unique solution \( f_j(x, \lambda) \) \( (f_j(x, \lambda)) \) belonging to \( L_2(-\infty, 0) \) \( (L_2(0, +\infty)) \). Therefore, the equation (1) has the solution belonging to \( L_2(-\infty, +\infty) \) if and only if \( \lambda \) is a root of the equation \( a_j(\lambda) = 0 \). Hence, any zero of \( a_j(\lambda) \) will be called an eigenvalue of the operator corresponding to the equation (1). From (26) we have that the function \( a_j(\lambda) \) has not any zero on \( \partial \Gamma_j^+ \) and therefore the zeros of the function \( a_j(\lambda) \) are placed in the interval \( (-|c_2|, |c_2|) \) or in the sheet \( \Gamma_j^+ \).

**Theorem 2:** The function \( a_j(\lambda) \) may have only a finite number of zeros.
**Proof:** We know that the zeros are placed in \((-|c_2|,|c_2|) \cup \Gamma_j^+\). Suppose that \(a_j(\lambda)\) has an infinite number of zeros \(\lambda^{(j)}_k\) \((k = 1, 2, \ldots)\). Since \(a_j(\lambda^{(j)}_k) = 0\) from (28) we obtain that the functions \(f_1(x, \lambda^{(j)}_k)\) and \(f_2(x, \lambda^{(j)}_k)\) are linearly dependent. Let us denote \(y^{(j)}_k(x) = f_1(x, \lambda^{(j)}_k)\) \((k = 1, 2, 3, \ldots)\) and let \(L^{(j)}_0\) to be the minimal closed operator generated on \(L_2(-\infty, +\infty)\) by differential expression \(-\frac{d^2}{dx^2} + q(x) - c_j^2\). It is clear that the operator \(L^{(j)}_0\) is selfadjoint. Since
\[
L^{(j)}_0 y^{(j)}_k(x) + 2\lambda^{(j)}_k p(x) y^{(j)}_k(x) = \left(\lambda^{(j)}_k\right)^2 - c_1^2 y^{(j)}_k(x)
\]
then
\[
\lambda^{(j)}_k = \left(\frac{py^{(j)}_k, y^{(j)}_k}{\|y^{(j)}_k\|^2}\right) \pm \sqrt{\left(\frac{py^{(j)}_k, y^{(j)}_k}{\|y^{(j)}_k\|^2}\right) + \left(L^{(j)}_0 y^{(j)}_k, y^{(j)}_k\right) + c_1^2 \left(y^{(j)}_k, y^{(j)}_k\right)},
\]
where \(\langle \cdot, \cdot \rangle\) is the inner product in \(L_2(-\infty, +\infty)\). Taking into account \(\lambda^{(j)}_k \in (-|c_2|, |c_2|) \cup \Gamma_j^+\) from (29) it is obtained that
\[
\left(L^{(j)}_0 y^{(j)}_k, y^{(j)}_k\right) < 0, \ k = 1, 2, \ldots
\]
Since \(y^{(j)}_k(x) = e^{k^{(j)}_j(\lambda^{(j)}_k)\cdot x}[1 + o(1)], x \to +\infty\) and the numbers \(\lambda^{(j)}_k\) are distinct from (30) we obtain that the selfadjoint operator \(L^{(j)}_0\) has an infinite number of negative eigenvalues. Therefore, our assumption is not true and \(a_j(\lambda)\) may have only a finite number of zeros.

**References**


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