A Uniqueness Theorem for Singular Sturm-Liouville Problem

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Abstract. In this paper, we show that if \( q(x) \) is prescribed on the \( \left( \pi/2, \pi \right) \) then the one spectrum suffices to determine \( q(x) \) on the interval \( (0, \pi/2) \). The potential function \( q(x) \) in a Sturm-Liouville problem is uniquely determined with one spectra by using the Hochstadt and Lieberman’s method [2].

Key Words: Sturm-Liouville problem, Spectrum

Introduction.

In this paper, we shall be concerned with an inverse Sturm-Liouville operator. We consider the operator

\[
Ly = -y'' + \left[ q(x) + \frac{v^2 - 1/4}{x^2} \right] y = \lambda y
\]

(1)
with the boundary conditions
\[ \lim_{x \to 0} \frac{y(x, \lambda)}{x^{1/2}} = \frac{1}{2^\nu \Gamma(\nu + 1)}, \quad (2) \]
\[ y(\pi, \lambda) \cos \beta + y'(\pi, \lambda) \sin \beta = 0. \quad (3) \]

The operator \( L \) is Self-Adjoint on the \( L^2[0, \pi] \) and with (2)–(3) boundary conditions has a discrete spectrum \( \{\lambda_n\} \). If condition (3) is replaced by
\[ y(\pi, \lambda) \cos \gamma + y'(\pi, \lambda) \sin \gamma = 0. \quad (4) \]
So, we obtain a new spectrum \( \{\lambda'_n\} \).

In this paper, we will consider a variation of the above inverse problem in that we will not require any information about a second spectrum but rather suppose \( q(x) \) is known almost everywhere on \( \left( \frac{\pi}{2}, \pi \right) \).

This information together with the spectrum \( \{\lambda_n\} \) of the problem (1)–(3) will be shown to determine \( q(x) \) uniquely on \( (0, \pi] \).

**Theorem**: We get the operator (1) with the boundary conditions (2) and (3). Let \( \{\lambda_n\} \) be the spectrum of \( L \) with (2) and (3). Consider a second operator
\[ \tilde{L} y = -y'' + \left[ \tilde{q}(x) + \frac{\nu^2 - 1/4}{x^2} \right] y = \lambda y \quad (5) \]
where \( \tilde{q}(x) \) is summable on the interval \( (0, \pi] \) and
\[ q(x) = \tilde{q}(x) \quad (6) \]
on the interval \( \left( \frac{\pi}{2}, \pi \right) \). Suppose that the spectrum of \( \tilde{L} \) with the (2)–(3) is also \( \{\lambda_n\} \).

Then \( q(x) = \tilde{q}(x) \) almost everywhere on \( (0, \pi] \).

**Proof**: Before proving the theorem we will first mention some results which will be need later. We take the following problems
\[ Ly = -y'' + \left[ q(x) + \frac{\nu^2 - 1/4}{x^2} \right] y = \lambda y \quad (7) \]
\[ \lim_{x \to 0} \frac{y(x, \lambda)}{x^{1/2}} = \frac{1}{2^\nu \Gamma(\nu + 1)} \quad (8) \]
and
\[ \ddot{y} = -y^* + \left[ \tilde{q}(x) + \frac{v^2 - 1/4}{x^2} \right] y = \lambda y \]  

(9)

\[ \lim_{x \to 0} \frac{\tilde{y}(x, \lambda)}{x^{\nu-1/2}} = \frac{1}{2^\nu \Gamma(\nu + 1)} \]  

(10)

As known [6], the Bessel’s functions of the first kind of order \( \nu \) is following asymptotic relations:

\[ J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[ \cos \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + O \left( \frac{1}{x} \right) \right], \]  

(11)

\[ J'_\nu(x) = -\sqrt{\frac{2}{\pi x}} \left[ \sin \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + O(1) \right]. \]  

(12)

It addition, It can be shown [5] that there exist a kernel \( H(x, t) \) continuous on \([0, \pi] \times [0, \pi]\) such that every solution of (7) and (8) can be expressed in the form

\[ y(x, \lambda) = \sqrt{x} \left( J_\nu \left( \sqrt{\lambda} x \right) + \int_0^x H(x, t) \sqrt{t} \left( J_\nu \left( \sqrt{\lambda} t \right) \right) dt \right) \]  

(13)

Where the kernel \( H(x, t) \) is solution of following problem

\[ \frac{\partial^2 H(x, t)}{\partial x^2} + \frac{v^2 - 1/4}{x^2} H(x, t) = \frac{\partial^2 H(x, t)}{\partial t^2} + \left[ \frac{v^2 - 1/4}{t^2} + q(t) \right] H(x, t), \]

\[ 2 \frac{dH(x, t)}{dx} = q(x), \]

\[ H(x, 0) = 0. \]

Analogous results to (13) hold for \( \tilde{y}(x, \lambda) \) in terms of a kernel \( \tilde{H}(x, t) \) which has similar properties of the \( H(x, t) \). Using equation (13) and Its for \( \tilde{y}(x, \lambda) \) we find that

\[ y \tilde{y} = \frac{x}{\sqrt{\lambda}} J'_\nu \left( \sqrt{\lambda} x \right) + \int_0^x \left[ H(x, t) + \tilde{H}(x, t) \right] \sqrt{t} \left( J_\nu \left( \sqrt{\lambda} t \right) \right) dt + \]

\[ \int_0^x H(x, t) \sqrt{t} J_\nu \left( \sqrt{\lambda} t \right) dt \times \int_0^s \tilde{H}(x, s) \sqrt{s} J_\nu \left( \sqrt{\lambda} s \right) ds. \]  

(14)

If the range of \( H(x, t) \) and \( \tilde{H}(x, t) \) is extended respect to the second argument and some straightforward computations, we rewrite (14) as
\[ y \tilde{y} = \frac{1}{2} \left\{ \frac{x}{(\sqrt{\lambda})} \left[ 1 + \cos 2 \left( \sqrt{\lambda} x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \right] \right\} ^{\frac{\pi}{x}} H(x, \tau) \cos 2 \left( \sqrt{\lambda} \tau - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) d\tau, \quad (15) \]

where

\[ H(x, \tau) = 2 \left[ H(x, x - 2\tau) + \tilde{H}(x, x - 2\tau) + \int_{x-2\tau}^{x} H(x, s) \tilde{H}(x, s - 2\tau) ds + \int_{-x}^{x-2\tau} H(x, s) \tilde{H}(x, s + 2\tau) ds \right]. \quad (16) \]

Now, we define the function

\[ \Omega(\lambda) = y(\pi, \lambda) \cos \beta + y'(\pi, \lambda) \sin \beta. \quad (17) \]

The zeros of \( \Omega(\lambda) \) are the eigenvalues of \( L \) or \( \tilde{L} \) subject to (2)-(3) and if the asymptotic results of \( y \) and \( y' \) are considered the \( \Omega(\lambda) \) is a entire function of order \( \frac{1}{2} \) of \( \lambda \).

If we multiply (7) by \( y' \) and (9) by \( y \) and subtract we obtain, after integration,

\[ (\tilde{y} y' - y \tilde{y}')|_0^\pi + \left[ (\tilde{q} - q) \tilde{y} \right] dx = 0. \quad (18) \]

Using (6) - (8) - (10), we obtain

\[ \left[ \tilde{y}(\pi, \lambda) y'(\pi, \lambda) - y(\pi, \lambda) \tilde{y}'(\pi, \lambda) \right]|_0^\pi + \left[ (\tilde{q} - q) \tilde{y} \right] dx = 0. \quad (19) \]

Now,

\[ Q = \tilde{q} - q \quad (20) \]

and

\[ K(\lambda) = \int_{0}^{\frac{\pi}{x}} Q(x) y \tilde{y} dx. \quad (21) \]

If the properties of \( y \) and \( \tilde{y} \) are considered, the function \( K(\lambda) \) is a entire function and for \( \lambda = \lambda_n \), since the first term of (19) is zero,

\[ K(\lambda_n) = 0. \quad (22) \]

In addition using (13) and (21) for \( 0 < x \leq \pi \),

\[ |K(\lambda)| \leq M \frac{1}{(\sqrt{\lambda})^{2\nu}}, \quad (23) \]
where \( M \) is constant. Now,

\[
\Psi(\lambda) = \frac{K(\lambda)}{\Omega(\lambda)},
\]

(24)

\( \Psi(\lambda) \) is a entire function. Asymptotic form of \( \Omega(\lambda) \) and with (23)

\[
|\Psi(\lambda)| = O\left(\frac{1}{\lambda^{1/2}}\right).
\]

So, From the Liouville Theorem for all \( \lambda \)

\[
\Psi(\lambda) = 0
\]

(25)

or

\[
K(\lambda) = 0.
\]

(26)

From now on, substituting (15) into (21)

\[
\frac{1}{2} \int_{0}^{\frac{\pi}{2}} Q(x) \left[ \frac{x}{(\sqrt{\lambda})^{\nu}} \left[ 1 + \cos 2\left( \sqrt{\lambda}x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \right] + \int_{0}^{\tau} \tilde{H}(x, \tau) \cos 2\left( \sqrt{\lambda} \tau - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) d\tau \right] dx = 0
\]

(27)

This can be written as

\[
\frac{x}{(\sqrt{\lambda})^{\nu}} \int_{0}^{\frac{\pi}{2}} Q(x) dx + \frac{\tau}{(\sqrt{\lambda})^{\nu}} \int_{0}^{\frac{\pi}{2}} \cos 2\left( \sqrt{\lambda} \tau - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \left[ Q(\tau) + \int_{\tau}^{\frac{\pi}{2}} Q(x) \tilde{H}(x, \tau) dx \right] d\tau = 0.
\]

(27)

Letting \( \lambda \to \infty \) for real \( \lambda \), we see from Riemann-Lebesque Lemma that we must have

\[
\int_{0}^{\frac{\pi}{2}} Q(x) dx = 0
\]

(29)

and

\[
\int_{0}^{\frac{\pi}{2}} \cos 2\left( \sqrt{\lambda} \tau - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \left[ Q(\tau) + \int_{\tau}^{\frac{\pi}{2}} Q(x) \tilde{H}(x, \tau) dx \right] d\tau = 0
\]

(30)

But from the completeness of the functions \( \cos \), we see that

\[
Q(\tau) + \int_{\tau}^{\frac{\pi}{2}} Q(x) \tilde{H}(x, \tau) dx = 0, \quad 0 < \tau < \frac{\pi}{2}
\]

(31)

Since equation (31) is a Volterra integral equations, it has only the zero solution. Hence
\[ q(x) = \tilde{q}(x) \]

almost everywhere.

References
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