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A Uniqueness the Theorem for Singular Sturm-Liouville Problem

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Abstract. In this paper, we show that If $q(x)$ is prescribed on the $(\pi/2, \pi]$ then the one spectrum suffices to determine $q(x)$ on the interval $(0, \pi/2)$.The potential function $q(x)$ in a Sturm Liouville problem is uniquely determined with one spectra by using the Hochstadt and Lieberman's method [2].
Key Words: Sturm-Liouville problem, Spectrum

Singüler Sturm-Liouville Problemi için Teklik Teoremi

Özet: Bu makalede gösterdi ki $q(x)$ $(\pi/2, \pi]$ aralığında tanımlanmış ise $(0, \pi/2)$ aralığı üzerinde $q(x)$ fonksiyonunu belirlemek için bir spektrum yeterlidir. Sturm-Liouville probleminde $q(x)$ potansiyel fonksiyonu Hochstadt ve Lieberman metodu kullanılarak bir spektruma göre tek olarak belirlenir.
Anahtar Kelimeler: Sturm-Liouville problem, Spectrum

Introduction.

In this paper, we shall be concerned with an inverse Sturm-Liouville operator. We consider the operator

$$Ly = -y'' + \left[q(x) + \frac{\nu^2 - 1/4}{x^2} \right] y = \lambda y \qquad (1)$$

with the boundary conditions

$$\lim_{x\to 0}\frac{y(x,\lambda)}{x^{\nu-1/2}}=\frac{1}{2^{\nu}\,\Gamma(\nu+1)}, \tag{2}$$

$$y(\pi,\lambda)\cos\beta+y'(\pi,\lambda)\sin\beta=0. \tag{3}$$

The operator L is Self-Adjoint on the $L_2[0,\pi]$ and with (2)–(3) boundary conditions has a discret spectrum $\{\lambda_n\}$. If condition (3) is replaced by

$$y(\pi,\lambda)\cos\gamma+y'(\pi,\lambda)\sin\gamma=0. \tag{4}$$

So, we obtain a new spectrum $\{\lambda'_n\}$.

In this paper, we will consider a variation of the above inverse problem in that we will not require any information about a second spectrum but rather suppose $q(x)$ is known almost everywhere on $\left(\frac{\pi}{2},\pi\right]$.

This information together with the spectrum $\{\lambda_n\}$ of the problem (1)–(3) will be shown to determine $q(x)$ uniquely on $(0,\pi]$.

Theorem : We get the operator (1) with the boundary conditions (2) and (3). Let $\{\lambda_n\}$ be the spectrum of L with (2) and (3). Consider a second operator

$$\tilde{L}y=-y''+\left[\tilde{q}(x)+\frac{\nu^2-1/4}{x^2}\right]y=\lambda y \tag{5}$$

where $\tilde{q}(x)$ is summable on the interval $(0,\pi]$ and

$$q(x)=\tilde{q}(x) \tag{6}$$

on the interval $\left(\frac{\pi}{2},\pi\right]$. Suppose that the spectrum of $\tilde{L}$ with the (2)–(3) is also $\{\lambda_n\}$. Then $q(x)=\tilde{q}(x)$ almost everywhere on $(0,\pi]$.

Proof : Before proving the theorem we will first mention some results which will be need later. We take the following problems

$$Ly=-y''+\left[q(x)+\frac{\nu^2-1/4}{x^2}\right]y=\lambda y \tag{7}$$

$$\lim_{x\to 0}\frac{y(x,\lambda)}{x^{\nu-1/2}}=\frac{1}{2^{\nu}\,\Gamma(\nu+1)} \tag{8}$$

and

$$\widetilde{L}y = -y'' + \left[\widetilde{q}(x) + \frac{\nu^2 - 1/4}{x^2}\right]y = \lambda y \tag{9}$$

$$\lim_{x \to 0} \frac{\widetilde{y}(x,\lambda)}{x^{\nu - 1/2}} = \frac{1}{2^\nu \Gamma(\nu + 1)} \tag{10}$$

As known [6], the Bessel's functions of the first kind of order ν is following asymptotic relations:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\cos\left[x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] + O\left(\frac{1}{x}\right)\right\}, \tag{11}$$

$$J'_\nu(x) = -\sqrt{\frac{2}{\pi x}}\left\{\sin\left[x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] + O(1)\right\}. \tag{12}$$

It addition , It can be shown [5] that there exist a kernel $H(x,t)$ continuous on $[0,\pi]\times[0,\pi]$ such that every solution of (7) and (8) can be expressed in the form

$$y(x,\lambda) = \frac{\sqrt{x}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}\, x\right) + \int_0^x H(x,t) \frac{\sqrt{t}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}\, t\right) dt \tag{13}$$

Where the kernel $H(x,t)$ is solution of following problem

$$\frac{\partial^2 H(x,t)}{\partial x^2} + \frac{\nu^2 - 1/4}{x^2} H(x,t) = \frac{\partial^2 H(x,t)}{\partial t^2} + \left[\frac{\nu^2 - 1/4}{t^2} + q(t)\right] H(x,t),$$

$$2\frac{dH(x,t)}{dx} = q(x),$$

$$H(x,0) = 0.$$

Analogous results to (13) hold for $\widetilde{y}(x,\lambda)$ in terms of a kernel $\widetilde{H}(x,t)$ which has similar properties of the $H(x,t)$. Using equation (13) and Its for $\widetilde{y}(x,\lambda)$ we find that

$$y\,\tilde{y} = \frac{x}{\left(\sqrt{\lambda}\right)^{2\nu}} J_\nu^2\left(\sqrt{\lambda}\, x\right) + \int_0^x \left[H(x,t) + \tilde{H}(x,t)\right] \frac{\sqrt{x\,t}}{\left(\sqrt{\lambda}\right)^{2\nu}} J_\nu\left(\sqrt{\lambda}\, x\right) J_\nu\left(\sqrt{\lambda}\, t\right) dt +$$

$$\int_0^x H(x,t) \frac{\sqrt{x}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}\, t\right) dt \times \int_0^x \tilde{H}(x,s) \frac{\sqrt{x}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}\, s\right) ds. \tag{14}$$

If the range of $H(x,t)$ and $\widetilde{H}(x,t)$ is extended respect to the second argument and some straightforward computations , we rewrite (14) as

$$y\,\tilde{y}=\frac{1}{2}\left\{\frac{x}{\left(\sqrt{\lambda}\right)^{2\nu}}\left[1+\cos 2\left(\sqrt{\lambda}x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right]\int_0^x \tilde{\tilde{H}}(x,\tau)\cos 2\left(\sqrt{\lambda}\tau-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)d\tau\right\}, \quad (15)$$

where

$$\tilde{\tilde{H}}(x,t)=2\left[H(x,x-2\tau)+\tilde{H}(x,x-2\tau)+\int_{-x+2\tau}^{x}H(x,s)\,\tilde{H}(x,s-2\tau)ds+\int_{-x}^{x-2\tau}H(x,s)\,\tilde{H}(x,s+2\tau)ds\right]$$

. (16)

Now, we define the function

$$\Omega(\lambda)=y(\pi,\lambda)\cos\beta+y'(\pi,\lambda)\sin\beta\,. \quad (17)$$

The zeros of $\Omega(\lambda)$ are the eigenvalues of L or $\tilde{L}$ subject to (2)-(3) and if the asymptotic results of y and y' are considered the $\Omega(\lambda)$ is a entire function of order $\frac{1}{2}$ of λ.

If we multiply (7) by y' and (9) by y and subtract we obtain , after integration ,

$$\left(\tilde{y}y'-y\tilde{y}'\right)\Big|_0^\pi+\int_0^x(\tilde{q}-q)y\tilde{y}dx=0\,. \quad (18)$$

Using (6) - (8) - (10) , we obtain

$$\left[\tilde{y}(\pi,\lambda)y'(\pi,\lambda)-y(\pi,\lambda)\tilde{y}'(\pi,\lambda)\right]_0^\pi+\int_0^{\frac{\pi}{2}}(\tilde{q}-q)dx=0\,. \quad (19)$$

Now ,

$$Q=\tilde{q}-q \quad (20)$$

and

$$K(\lambda)=\int_0^{\frac{\pi}{2}}Q(x)y\tilde{y}dx\,. \quad (21)$$

If the properties of y and $\tilde{y}$ are considered , the function $K(\lambda)$ is a entire function and for $\lambda=\lambda_n$, since the first term of (19) is zero ,

$$K(\lambda_n)=0\,. \quad (22)$$

In addition using (13) and (21) for $0<x\le\pi$,

$$\left|K(\lambda)\right|\le M\frac{1}{\left(\sqrt{\lambda}\right)^{2\nu}}, \quad (23)$$

where M is constant. Now ,

$$\Psi(\lambda)=\frac{K(\lambda)}{\Omega(\lambda)}, \tag{24}$$

$\Psi(\lambda)$ is a entire function. Asymptotic form of $\Omega(\lambda)$ and with (23)

$$|\Psi(\lambda)|=O\left(\frac{1}{\lambda^{\nu+\frac{1}{2}}}\right).$$

So , From the *Liouville Theorem* for all λ

$$\Psi(\lambda)=0 \tag{25}$$

or

$$K(\lambda)=0\,. \tag{26}$$

From now on , substituting (15) into (21)

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} Q(x)\left\{\frac{x}{\left(\sqrt{\lambda}\right)^{2\nu}}\left[1+Cos2\left(\sqrt{\lambda}x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right]+\int_0^x \tilde{\tilde{H}}(x,\tau)Cos2\left(\sqrt{\lambda}\tau-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)d\tau\right\}dx=0$$

. (27)

This can be written as

$$\frac{x}{\left(\sqrt{\lambda}\right)^{2\nu}}\int_0^{\frac{\pi}{2}} Q(x)dx+\frac{\tau}{\left(\sqrt{\lambda}\right)^{2\nu}}\int_0^{\frac{\pi}{2}} Cos2\left(\sqrt{\lambda}\tau-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left[Q(\tau)+\int_\tau^{\frac{\pi}{2}} Q(x)\tilde{\tilde{H}}(x,\tau)dx\right]d\tau=0\,.$$

Letting $\lambda\to\infty$ for real λ , we see from *Riemann -Lebesque Lemma* that we must have

$$\int_0^{\frac{\pi}{2}} Q(x)dx=0 \tag{29}$$

and

$$\int_0^{\frac{\pi}{2}} Cos2\left(\sqrt{\lambda}\tau-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left[Q(\tau)+\int_\tau^{\frac{\pi}{2}} Q(x)\tilde{\tilde{H}}(x,\tau)dx\right]d\tau=0 \tag{30}$$

But from the completeness of the functions *Cos* , we see that

$$Q(\tau)+\int_\tau^{\frac{\pi}{2}} Q(x)\tilde{\tilde{H}}(x,\tau)dx=0,\quad 0<\tau<\frac{\pi}{2} \tag{31}$$

Since equation (31) is a Volterra integral equations, it has only the zero solution. Hence

$$q(x) = \tilde{q}(x)$$

almost everywhere.

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